

An impulsive dynamical systems framework for reset control systems

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Abstract

Impulsive dynamical systems is a well-established area of dynamical systems theory, and it is used in this work to analyze several basic properties of reset control systems: existence and uniqueness of solutions, and continuous dependence on the initial condition (well-posedness). The work scope is about reset control systems with a linear and time-invariant base system, and a zero-crossing resetting law. A necessary and sufficient condition for existence and uniqueness of solutions, based on the well-posedness of reset instants, is developed. As a result, it is shown that reset control systems (with strictly proper plants) do not have Zeno solutions. It is also shown that full reset and partial reset (with a special structure) always produce well-posed reset instants. Moreover, a definition of continuous dependence on the initial condition is developed, and also a sufficient condition for reset control systems to satisfy that property. Finally, this property is used to analyze sensitivity of reset control systems to sensor noise. This work also includes a number of illustrative examples motivating the key concepts and main results.

Index Terms

Reset control systems, impulsive dynamical systems, hybrid systems.

I. INTRODUCTION

Reset control systems trace back to the seminal work of Clegg [21], that introduced a nonlinear integrator that sets its output to zero whenever its input is zero. Almost two decades later, the works by Horowitz and coworkers ([30], [31]) propose design methods to incorporate a Clegg integrator (CI), and also a first order reset element (FORE), into a control loop. In the late 90s, the term *reset controller* is finally coined in the works by Holot, Chait and coworkers ([14]), to describe a 'linear and time invariant system with mechanisms and laws to reset their states to zero', being the main motivation its use for overcoming fundamental limitations of linear and time invariant (LTI) control systems.

Impulsive and hybrid systems are active areas of dynamical systems theory that have been developed in the last three decades ([4], [5], [12], [26], [28], [36]–[38], [40], [49]). Since reset controller dynamics is a combination of time and event based dynamics, it is not surprising that in the last decade different impulsive/hybrid dynamical system formulations were used for modeling and analysis of reset control systems. The survey [44] emphasizes the diversity of hybrid systems formulations: hybrid automata, switched systems, piecewise models, complementary systems, hybrid inclusions, There are two main frameworks that have been successfully used for modeling reset control systems: the framework of impulsive dynamical systems (IDS) [28], used in [9] and references therein; and the framework of hybrid inclusions (HI) developed in [26], used in [1], [41], [42]. Finally, another formulation of reset systems as hybrid automata has been investigated in [43].

From a control practice point of view, an important issue in the different impulsive/hybrid systems formulations, directly related with their solution concept, is well-posedness. Historically, the term *well-posedness* comes from Hadamard [34], who believed that mathematical models of physical phenomena should have these properties: i) a solution exists, ii) the solution is unique, and iii) the solution depends

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continuously on the initial condition (and, in general, on the problem data). Regarding impulsive/hybrid systems, well-posedness (in the sense of Hadamard) has been recognized to be a very hard issue, and the term has been relaxed in different ways. In [17], [32], [35], [38], well-posedness is directly based on the existence and uniqueness of solutions, while in [26] uniqueness of solutions is excluded and well-posedness is restricted to a relaxed sense of continuous dependence on the initial condition. On the other hand, although the term well-posedness is not explicitly used, existence and uniqueness of solutions, and also continuous dependence on the initial condition, are a main issue in the IDS framework [5], [28], [36], [40].

In this work, both existence and uniqueness of solutions and continuous dependence on the initial condition will be investigated for reset control systems in the IDS framework, taking as a starting point the classical zero-crossing resetting law of Clegg and Horowitz. This formulation has been followed in several other recent works, for example [13], [14] and references therein, and also [6], [7], [9], [10], [19], [20], [47], including some successful experimental applications.

In these precedent works, *existence and unicity of solutions* is simply assumed or is overtaken by using time regularization, that is modifying the resetting law definition by allowing reset actions to be performed only if some finite time has passed since the last reset action. However, the problem is simply avoided and thus an in-depth analysis of the problem is missed. In addition, it is generally assumed that time regularization poses intrinsic difficulties in analysis and implementation of reset control systems [29], [41], and thus it is desirable to remove that restriction. To the knowledge of authors, the first work about existence and uniqueness of solutions, without including time regularization, is [8]. A related recent work considers reset systems with a different resetting law, based on a reset band that includes a type of spatial regularization [11]. In [8], a sufficient condition is developed, given by the non-existence of after-reset states that are elements of the unobservable subspace of the base system. This work will follow this research direction to investigate existence and uniqueness of solutions, searching necessary and sufficient conditions.

In addition, the IDS framework will also be used for investigating *continuous dependence of solutions on the initial condition*, and in contrast with the HI framework, without losing uniqueness of solutions. It will be shown how for reset control systems, with a base LTI system, and with exogenous signals modeled by Bohl functions, continuous dependence on the initial condition can be characterized without introducing nondeterminism. In general, pointwise continuous dependence on the initial condition is not a common property of impulsive and hybrid systems [26], [28], [38]. This is mainly due to the fact that for two solutions corresponding to an initial condition and some small perturbation on it, the discontinuity instants are in general different. More specifically, in the IDS framework, a quasi-continuous property has been introduced in [28], that is based on the continuity of the maps relating the initial condition to the resetting instants, but it has been shown that this property is not satisfied for reset control systems [8], [9]. This fact makes the continuous dependence problem challenging, and in fact to the knowledge of authors it is unexplored for reset control systems in the IDS framework. This has been a main motivation for this work, where a notion of continuous dependence inspired in [23], that uses the Hausdorff metric, will be used for the characterization of this fundamental property. Some preliminary related recent work, developed in the HI framework appears in [22]. Here, it should be emphasized that the well-posedness concept of the HI framework (see [27], Ch. 6, p. 126) is based on a relaxed sense of continuous dependence (outer semicontinuous dependence to be precise), and should not be confused with the continuous dependence concept to be developed in this work, which is a much stronger property.

Summarizing, this work will elaborate a rigorous IDS framework for reset control systems, approaching two basic problems regarding well-posedness: existence and uniqueness of solutions, and continuous dependence on the initial condition. Although it is only investigated the zero-crossing resetting law, it is believed that the different concepts and methods to be developed will provide a solid framework to analyze most of the resetting laws that has been found useful in practice. The main contributions of this work are:

- Existence and uniqueness of reset control systems solutions on forward time (excluding pathological

behaviors like deadlock and existence of Zeno solutions), is shown to be equivalent to the well-posedness of reset instants (they are well defined and distinct).

- For reset systems, not necessarily reset control systems, well-posedness of reset instants is shown to be equivalent to the invariance of a subspace which is a subset of the base system unobservable subspace.
- For the significative class of reset compensators with full reset, reset control systems have always well-posed reset instants, as far as the exogenous inputs are generated by exosystems (Bohl functions). In the case of reset compensation with partial reset, reset control systems are guarantied to have well-posed reset instants only in some cases: the reset compensator has a special structure, or some zero/pole cancellations of a particular structure are present. As a result, the time regularization restriction may be removed in the reset compensator definition, since it is not necessary for avoiding Zeno solutions and deadlock.
- A sufficient condition for continuous dependence on the initial condition, using a new elaborated concept based on the Hausdorff distance between trajectories.
- An analysis of reset control system sensitivity to sensor noise based on the developed property of continuous dependence on the initial condition. Again, it is shown how full reset/partial reset compensators produce reset control systems that are not sensitive to sensor noise, where the noise signal is an arbitrary Bohl function.

In Section II, besides notation, IDS and also reset control system are formally defined, the solution concept is elaborated and some basic properties are also stated. Section III is devoted to the existence and uniqueness of solutions for reset systems, based on the equivalent property of reset instants well-posedness. In addition, necessary and sufficient conditions are developed for well-posedness of reset instants; it is also shown how reset control systems based on full reset compensators (or partial reset with a particular structure -right reset-) always have well-posed reset instants. In Section IV, a concept of continuous dependence is developed based on the Hausdorff distance between a trajectory and a perturbed trajectory. It is shown with some simple counterexamples that functions mapping initial conditions to reset instants have jump discontinuities, and thus previous IDS continuous dependence results are useless to approach the problem. Finally, a sufficient condition for continuous dependence of reset control systems on the initial condition is given, including several illustrative examples; moreover, for a class of sensor noise signals, modeled as Bohl functions, the characterization of sensitivity with respect to sensor noise is also investigated using the continuous dependence property.

II. IMPULSIVE DYNAMICAL SYSTEMS AND RESET SYSTEMS

A. Notation and Background

\mathbb{N} is the natural numbers set, \mathbb{R}^+ is the set of nonnegative real numbers, \mathbb{C} is the complex numbers set, \mathbb{R}^n is the n -dimensional euclidean space, and (\mathbf{x}, \mathbf{y}) , with column vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, denotes the column vector $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$. $(t_k)_{k=1,2,\dots}$ is a sequence of real numbers. I_n is the $n \times n$ identity matrix, $O_{n_1 \times n_2}$ is the $n_1 \times n_2$ zero matrix (if it is clear from the context subscripts are eliminated), and $\mathbf{0}$ is a column vector of zeros. $\mathcal{N}(A)$, for a matrix $A \in \mathbb{R}^{n \times m}$, stands for the null space of A . \emptyset denotes the empty set, and \setminus denotes sets difference; when used with a sequence, $\mathbb{R} \setminus (t_k)_{k=1,2,\dots} := \mathbb{R} \setminus \{t_1, t_2, \dots\}$. For a set $\mathcal{X} \subset \mathbb{R}$, $\min \mathcal{X}$ is an element of \mathcal{X} that is its greatest lower bound, and $\min \emptyset = \infty$. For a set $\mathcal{X} \subset \mathbb{R}^n$, $\bar{\mathcal{X}}$ denotes its closure. \mathcal{S}^{n-1} is a $(n-1)$ -sphere, $\mathcal{S}^{n-1} := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$; \mathcal{HS}^{n-1} is a unit $(n-1)$ -hemisphere, $\mathcal{HS}^{n-1} := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 = \cos(\theta_1), x_2 = \sin(\theta_1) \cos(\theta_2), \dots, x_n = \sin(\theta_1) \sin(\theta_2) \dots \sin(\theta_{n-1}); \theta_1, \dots, \theta_{n-2} \in [0, \pi], \theta_{n-1} \in [0, \pi)\}$. \leftarrow is an assignment. \vee and \wedge are the logical disjunction and conjunction, respectively.

A Bohl function ([1], [46]) is defined as a linear combination of functions of the form $t^k e^{\lambda t}$, where k is a nonnegative integer and $\lambda \in \mathbb{C}$. Given a matrix $A \in \mathbb{R}^{n \times n}$ and a linear subspace $\mathcal{V} \subset \mathbb{R}^n$, \mathcal{V} is A -invariant ([25]) if $A\mathbf{x} \in \mathcal{V}$ for any $\mathbf{x} \in \mathcal{V}$. With some abuse of notation, $I := [0, T] \subseteq \mathbb{R}^+$ is an interval

where the endpoint T may be finite or infinite (if the interval is related with data like an initial condition \mathbf{x}_0 , $I_{\mathbf{x}_0}$ is used). A function $f : I \rightarrow \mathbb{R}^n$ is *left continuous with right limits* (or simply *left-continuous*) in I if the left limit $\mathbf{x}(a^-) := \lim_{t \rightarrow a, t < a} \mathbf{x}(t)$ exists and $\mathbf{x}(a) = \mathbf{x}(a^-)$ for any $a \in I \setminus \{0\}$, and the right limit $\mathbf{x}(a^+) := \lim_{t \rightarrow a, t > a} \mathbf{x}(t)$ exists for any $a \in I$.

A (linear) state-dependent impulsive dynamical system (IDS) is given by

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t), & \mathbf{x}(t) \notin \mathcal{M} \\ \mathbf{x}(t^+) = A_R\mathbf{x}(t), & \mathbf{x}(t) \in \mathcal{M} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $t \geq 0$, is the system state at the instant t , $\mathcal{M} \subset \mathbb{R}^n$ is the reset set, and $A, A_R \in \mathbb{R}^{n \times n}$. State-dependent IDS has been developed in the monograph [28] and references therein, and will be used in this work as the framework to represent reset control systems and to investigate well-posedness.

The first equation in (1) will be referred to as the *base system*, while the second equation in (1) will be referred to as the *resetting law*. For this IDS, there exists a unique solution $\psi(t) = e^{At}\psi_0$ of the (continuous) base system with initial condition $\psi(0) = \psi_0$ on $[0, \infty)$, for any $\psi_0 \in \mathbb{R}^n$. When at some instant $t \geq 0$, referred to as *reset instant*, $\mathbf{x}(t) \in \mathcal{M}$ is true (a *crossing* is performed) the state $\mathbf{x}(t)$ jumps to $\mathbf{x}(t^+) = A_R\mathbf{x}(t) \in \mathcal{M}_R$, where $\mathcal{M}_R := A(\mathcal{M})$ is the after-reset set. Otherwise, the state $\mathbf{x}(t)$ evolves with the base system dynamics. Here, the term *crossing* is used in a relaxed sense, it has to be understood that a crossing is performed when the solution intersects the reset set.

For a given initial condition \mathbf{x}_0 , reset instants are denoted by t_k , $k = 1, 2, \dots$. A function $\mathbf{x} : I_{\mathbf{x}_0} \rightarrow \mathbb{R}^n$ is a *solution* of the IDS (1) on the interval $I_{\mathbf{x}_0}$, with initial condition \mathbf{x}_0 , if ([5], [28], [36])

- $\mathbf{x}(0) = \mathbf{x}_0$
- \mathbf{x} is differentiable and $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ for any $t \in I_{\mathbf{x}_0}$, $t > 0$, $t \neq t_k$, $k = 1, 2, \dots$
- $\mathbf{x}(t)$ is left-continuous in $I_{\mathbf{x}_0}$ and $\mathbf{x}(t^+) = A_R\mathbf{x}(t)$ if $t = t_k \in I_{\mathbf{x}_0}$, $k = 1, 2, \dots$

Note that for a particular solution there may exist no crossings, a finite or a infinite number of crossings, and in a finite or infinite time interval $I_{\mathbf{x}_0}$. And that, if a crossing is performed at the instant $a \in I_{\mathbf{x}_0}$, then the solution has a jump discontinuity at that instant, that is the limits $\mathbf{x}(a^+)$ and $\mathbf{x}(a^-)$ exist, and $\mathbf{x}(a) = \mathbf{x}(a^-)$. A IDS solution is a *càglàd* function (French "continue à gauche limite à droite"), not to be confused with *càdlàg* functions ("continue à droite limite à gauche"), mentioned for example in [26].

On the other hand, note that if the sets \mathcal{M} and \mathcal{M}_R are not disjoint, then for $\mathbf{x}_0 \in \mathcal{M}_R \cap \mathcal{M}$ an infinity number of jumps are performed without continuous evolution between them (this is usually referred to as livelock [28]) and no solution of the IDS (1) would exist according to the above definition. Thus, the following standing assumption must be satisfied: $\mathcal{M}_R \cap \mathcal{M} = \emptyset$.

B. Reset systems

In this work, reset systems are defined as a particular class of the IDS (1), in which the resetting law is based on the crossing of an hyperplane given by $\mathcal{H}_C := \{\mathbf{x} \in \mathbb{R}^n : C\mathbf{x} = 0\}$, that is $\mathcal{H}_C = \mathcal{N}(C)$, where C is a row vector; and, in addition, A_R is an orthogonal projector such as the last n_r components of $A_R\mathbf{x}$ are zero, and the first $\bar{n}_r := n - n_r$ components of $A_R\mathbf{x}$ remain unchanged after a crossing at $\mathbf{x} \in \mathcal{M}$. Note that the subspace \mathcal{H}_C can not be directly used as the reset set, at least the origin $\mathbf{0}$ belongs both to \mathcal{H}_C and $A_R(\mathcal{H}_C)$, and thus it always true that $\mathcal{H}_C \cap A_R(\mathcal{H}_C) \neq \emptyset$. Thus, for building the reset set from \mathcal{H}_C , it is necessary to remove from \mathcal{H}_C all the fixed points of A_R as a map. In the following, this set of fixed points will be denoted by \mathcal{F}_R .

Definition II.1 (reset system): A reset system (A, C, n_r) , where $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$, $n \geq 2$, and $1 \leq n_r < n$, is an IDS as given by (1), with A_R given by

$$A_R = \begin{pmatrix} I_{\bar{n}_r} & O_{\bar{n}_r \times n_r} \\ O_{n_r \times \bar{n}_r} & O_{n_r \times n_r} \end{pmatrix} \quad (2)$$

and reset set \mathcal{M} given by

$$\mathcal{M} = \mathcal{N}(C) \setminus \mathcal{F}_R \quad (3)$$

where $\mathcal{F}_R = \mathcal{N}\left(\begin{pmatrix} I - A_R \\ C \end{pmatrix}\right)$ and $n_{\bar{r}} = n - n_r$.

Note that $n_r = n$ would be the case in which the state is fully reset to zero at a crossing; therefore, either the reset system evolves as the base system if no crossings are performed, or the system reaches the origin at the first crossing. This trivial case has been removed from the above definition, and thus no first order reset systems may exist. In addition, (3) is consistent with [14], that is a reset is performed at the instant t if $C\mathbf{x}(t) = 0$ and $(I - A_R)\mathbf{x}(t) \neq 0$.

It is also convenient to introduce the hyperplane $\mathcal{H}_R := \mathcal{N}(I - A_R)$. Note that the definition of the reset set \mathcal{M} , according to (3), only depends on n_r and C . Since A_R is a projector, then $\mathcal{F}_R = \mathcal{N}(I - A_R) \cap \mathcal{N}(C) = \mathcal{H}_C \cap \mathcal{H}_R$; and thus \mathcal{M} contains all the states of the hyperplane \mathcal{H}_C that are not fixed points of A_R . Therefore, it is clear that the after-reset set $\mathcal{M}_R = A_R(\mathcal{M})$ does not contain states of the reset set \mathcal{M} , and thus $\mathcal{M}_R \cap \mathcal{M} = \emptyset$. In addition, note that the set of fixed points \mathcal{F}_R is a subspace while the reset set \mathcal{M} is not ($\mathbf{0} \notin \mathcal{M}$), and that $\bar{\mathcal{M}} = \mathcal{M} \cup \mathcal{F}_R = \mathcal{H}_C$. On the other hand, for reset systems with $C = (C_{n_{\bar{r}}}, O_{1 \times n_r})$ (for example the reset control systems in Section III.B), it results that $C(I - A_R) = O$ (that is the hyperplanes \mathcal{H}_C and \mathcal{H}_R are orthogonal), and thus it easily follows that $\mathcal{M}_R = \mathcal{F}_R$. Some examples:

1) *Second order reset systems ($n = 2$):* note that the simplest reset system is second order; for a second order reset system $(A, C, 1)$ ($n_r = 1$ is the only possible value), three subclasses are possible (assuming that $C \neq O$):

- (Fig. 1.a) If $C = \begin{pmatrix} c_1 & 0 \end{pmatrix}$ for some $c_1 \in \mathbb{R} \setminus \{0\}$, then $\mathcal{H}_C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$, $\mathcal{M} = \mathcal{H}_C \setminus \{\mathbf{0}\}$, and $\mathcal{M}_R = \mathcal{F}_R = \mathcal{H}_R = \{\mathbf{0}\}$,
- If $C = \begin{pmatrix} 0 & c_2 \end{pmatrix}$ for some $c_2 \in \mathbb{R} \setminus \{0\}$, then $\mathcal{H}_C = \mathcal{H}_R = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, $\mathcal{F}_R = \mathcal{H}_C$, and $\mathcal{M} = \mathcal{M}_R = \emptyset$ (this is a trivial case, no reset action may be performed since $\mathcal{M} = \emptyset$),
- (Fig. 1.b) Otherwise, $\mathcal{H}_C = \mathcal{N}(C)$, $\mathcal{F}_R = \{\mathbf{0}\}$, $\mathcal{M} = \mathcal{H}_C \setminus \{\mathbf{0}\}$, $\mathcal{H}_R = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, and $\mathcal{M}_R = \mathcal{H}_R \setminus \{\mathbf{0}\}$.

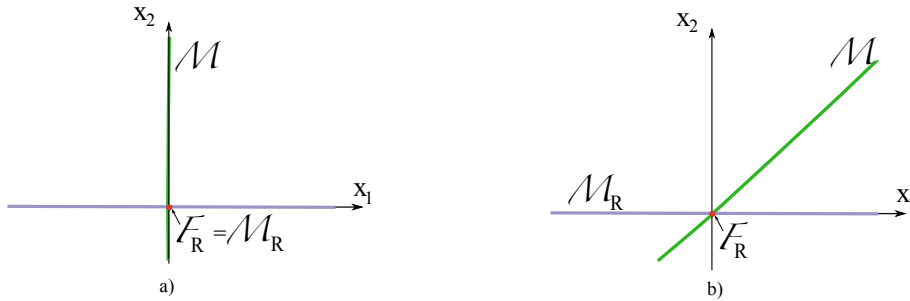


Fig. 1. \mathcal{M} , \mathcal{M}_R , and \mathcal{F}_R for second order reset systems

2) *Third order reset systems ($n=3$):* In this case, $n_r \in \{1, 2\}$. For $n_r = 1$, the hyperplane \mathcal{H}_R is the plane $x_1 - x_2$, and several subclasses are possible depending on whether the hyperplanes \mathcal{H}_C and \mathcal{H}_R are orthogonal or not. Assume that they are not identical (if they are identical then a trivial case with $\mathcal{M} = \emptyset$ is obtained), then $\mathcal{M}_R = \mathcal{F}_R$ if they are orthogonal (Fig. 2.b), and $\mathcal{M}_R = \mathcal{H}_R \setminus \mathcal{F}_R$, otherwise (Fig. 2.a). For $n_r = 2$ the hyperplane \mathcal{H}_R is the x_1 -axis, and again several subclasses are possible, for example Fig. 2.c shows the case corresponding to $\mathcal{H}_C \cap \mathcal{H}_R = \{\mathbf{0}\}$, and then $\mathcal{F}_R = \{\mathbf{0}\}$, $\mathcal{M} = \mathcal{H}_C \setminus \{\mathbf{0}\}$, and $\mathcal{M}_R = \mathcal{H}_R \setminus \{\mathbf{0}\}$.

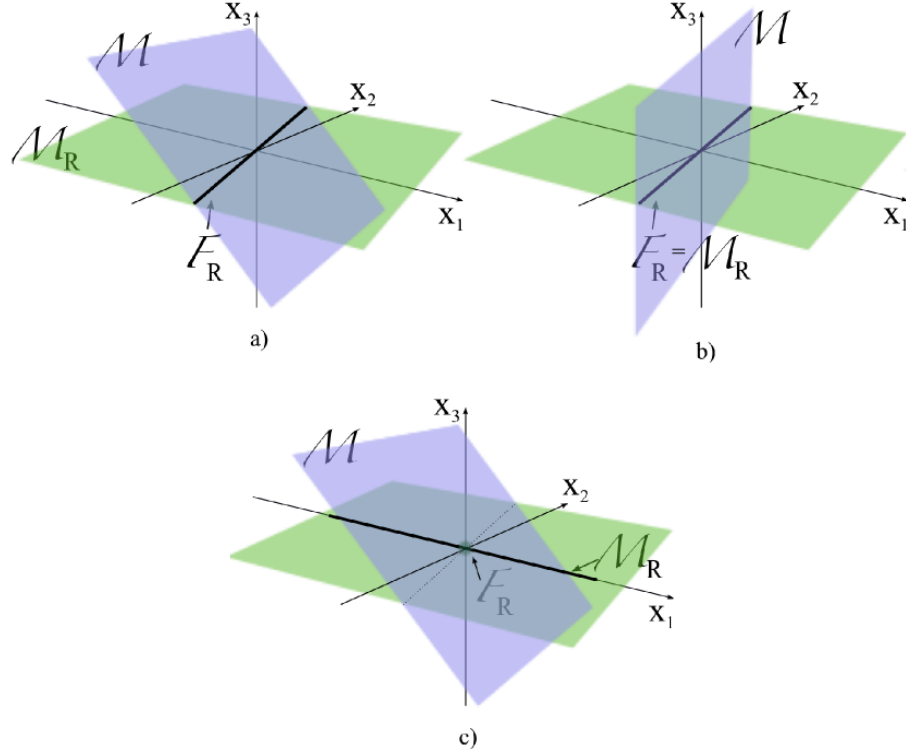


Fig. 2. \mathcal{M} , \mathcal{M}_R , and \mathcal{F}_R for third order reset systems

3) *Reset systems with $C = (C_{n_r}, O_{n_r})$* : this is an important case, corresponding for example to the reset control systems to be analyzed in Section IV. In this case, $C(I - A_R) = O$ (that is the hyperplanes \mathcal{H}_C and \mathcal{H}_R are orthogonal), and thus it easily follows that $\mathcal{M}_R = \mathcal{F}_R$ (see Fig. 1.a and Fig. 2.b).

III. EXISTENCE AND UNIQUENESS OF SOLUTIONS

A key topic in IDS analysis is the existence and uniqueness of solutions. In this Section, the concept of reset instants well-posedness will be elaborated, and it will be shown to be equivalent to the existence and uniqueness of reset control systems solutions on forward time, and for any arbitrary initial condition.

A. Reset systems with well-posed reset instants

Definition III.1 (well-posed reset instants): A reset system (A, C, n_r) has well-posed reset instants if for any initial condition $\mathbf{x}_0 \in \mathbb{R}^n$ there exists a sequence $\mathbb{T} : \{1, 2, \dots\} \rightarrow \mathbb{R}^+ \cup \{\infty\}$, denoted by $\mathbb{T} = (t_1, t_2, \dots)$, and given by the following procedure, being $t_0 = 0$:

- $t_1 = \begin{cases} 0 & , \mathbf{x}_0 \in \mathcal{M} \\ \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\} & , \mathbf{x}_0 \in \mathbb{R}^n \setminus \mathcal{M} \end{cases}$
- $i = 1$
while $t_i \neq \infty$ **do**
 $\mathbf{x}_i = A_R e^{A(t_i - t_{i-1})} \mathbf{x}_{i-1}$
 $t_{i+1} = t_i + \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_i \in \mathcal{M}\}$
 $i \leftarrow i + 1$
 $N = i$
end while

Here \mathbb{T} is the finite or infinite sequence of reset instants corresponding to an initial condition. Note that, for a reset system with well-posed reset instants, all the reset instants are distinct, since for $i \geq 1$ after-reset states \mathbf{x}_i satisfy $\mathbf{x}_i = A_R e^{A(t_i - t_{i-1})} \mathbf{x}_{i-1} \notin \mathcal{M}$ (\mathcal{M} and \mathcal{M}_R are disjoint). That is, $\mathbb{T} = (t_1, t_2, \dots)$

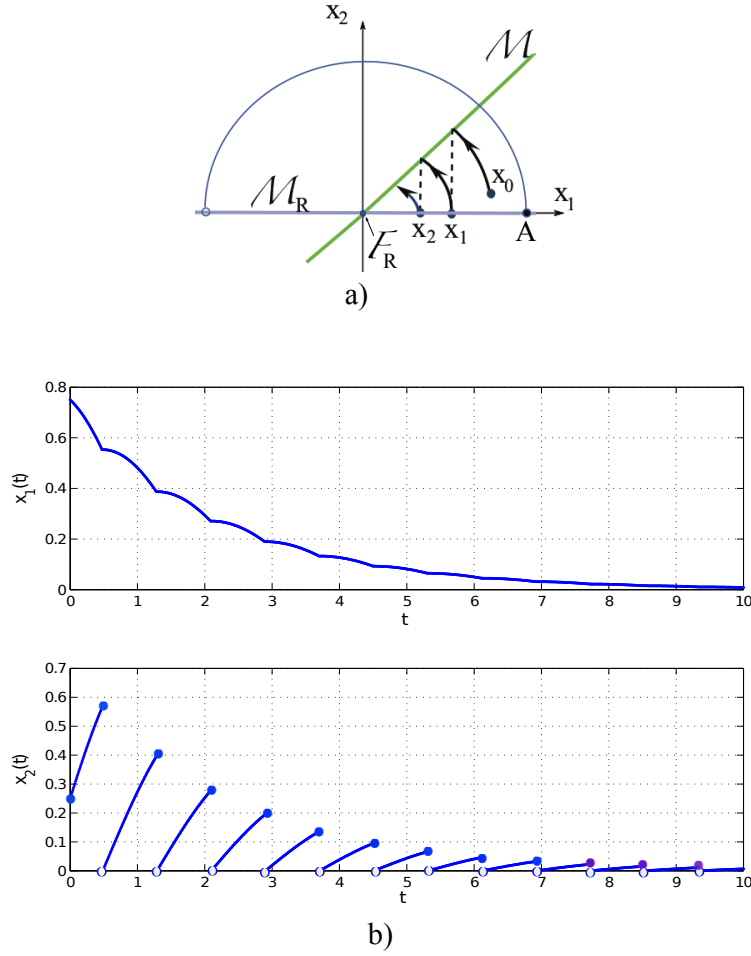


Fig. 3. a) \mathcal{M} , \mathcal{M}_R , and \mathcal{F}_R for Example III.1, b) Time simulation for $\omega = 1$ and $\mathbf{x}_0 = (0.75, 0.25)$

satisfies $0 \leq t_1 < t_2 < \dots < \infty$ for any initial condition, the three possible cases are: i) $\mathbb{T} = (\infty)$ (there is no reset actions), ii) $\mathbb{T} = (t_i)_{i=1}^N = (t_1, t_2, \dots, t_{N-1}, \infty)$ (a finite number, $N - 1$, of reset actions), and iii) $\mathbb{T} = (t_i)_{i=1}^\infty$ (an infinity number of reset actions). On the other hand, a system that does not have well-posed reset instants is said to have *ill-posed* reset instants.

Example III.1 (reset system with well-posed reset instants): Consider a second order reset system $(A, C, 1)$ with

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 \end{pmatrix} \quad (4)$$

for some constant $\omega > 0$. By definition, \mathcal{H}_R is the x_1 -axis, $\mathcal{H}_C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$, $\mathcal{F}_R = \{\mathbf{0}\}$, $\mathcal{M} = \mathcal{H}_C \setminus \{\mathbf{0}\}$, and $\mathcal{M}_R = \mathcal{H}_R \setminus \{\mathbf{0}\}$ (see Fig. 1.a). It is not difficult to see that this reset system has reset instants given by

- $t_1 = \begin{cases} \infty & x_{20} = x_{10} = 0 \\ \frac{1}{\omega}(\frac{\pi}{4} - \text{atan}(\frac{x_{20}}{x_{10}})) & \text{otherwise} \end{cases}$
- For $i \geq 2$, $t_i = t_{i-1} + \frac{\pi}{4\omega}$

That is, $\mathbb{T} = (\infty)$ for $\mathbf{x}_0 = \mathbf{0}$ (since $\mathbf{0} \notin \mathcal{M}$ then no crossings are performed), and $\mathbb{T} = (\frac{1}{\omega}(\frac{\pi}{4} - \text{atan}(\frac{x_{20}}{x_{10}})) + (i-1)\frac{1}{2}\frac{\pi}{4\omega})_{i=1}^\infty$ for any $\mathbf{x}_0 \neq \mathbf{0}$ (a periodic sequence after the second reset instant, with fundamental period $\frac{\pi}{4\omega}$). Fig. 3 shows a solution of the reset system with initial condition \mathbf{x}_0 , and after-reset states $\mathbf{x}_1, \mathbf{x}_2, \dots$.

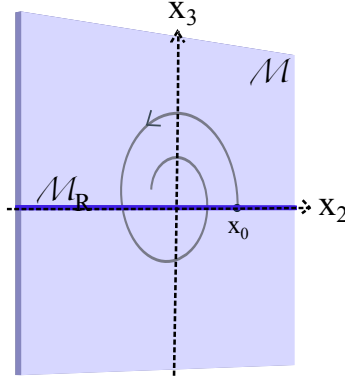


Fig. 4. Reset system with ill-posed reset instants (it is shown the trajectory of the base system): $\mathbf{x}(\epsilon) \in \mathcal{M}$ for any $\mathbf{x}_0 \in \mathcal{M}_R$ and $\epsilon \in (0, t^*)$.

Example III.2 (reset system with ill-posed reset instants): This system is used in [41] for analyzing some weak points in the definition of reset systems given in [14]. Consider a reset system $(A, C, 1)$ with

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad (5)$$

where \mathcal{H}_R is the $x_1 - x_2$ plane, \mathcal{H}_C is the $x_2 - x_3$ plane, $\mathcal{F}_R = \mathcal{M}_R = \text{span}\{(0, 1, 0)\}$ (the x_2 -axis), and $\mathcal{M} = \mathcal{H}_C \setminus \mathcal{F}_R$ (see Fig. 2). In [41] it is correctly argued that for any initial condition $\mathbf{x}_0 = (0, a, 0) \in \mathcal{M}_R$, the solution is ill-defined. In fact, the problem is that the reset system has ill-posed reset instants, since for any $\mathbf{x}_0 = (0, a, 0) \in \mathcal{M}_R$, with $a \neq 0$,

$$\min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\} = \min(0, T_1)$$

does not exist since the interval $(0, T_1)$ is open and thus t_1 (and \mathbb{T}) does not exist. Here $T_1 > 0$ is some instant prior to $t^* \approx 3.11$, the first non-zero instant in which $e^{At^*}\mathbf{x}_0 \in \mathcal{M}_R$. Note that the trajectory of the base system is a stable focus in the $x_2 - x_3$ plane (Fig. 2).

Proposition III.1: *The reset system (A, C, n_r) has well-posed reset instants if and only if for any $\mathbf{x}_0 \in \mathcal{HS}^{n-1} \cap \mathcal{F}_R$ there exists a number $t_1 \in \mathbb{R}^+ \cup \{\infty\}$ such that*

$$t_1 = \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\} \quad (6)$$

Proof: Since by definition $\mathcal{F}_R = \mathcal{H}_C \cap \mathcal{H}_R$ and $\mathcal{M} = \mathcal{H}_C \setminus \mathcal{F}_R$, then $\bar{\mathcal{M}} \setminus \mathcal{M} = \mathcal{F}_R$. For $\mathbf{x}_0 = \mathbf{0}$ ($\notin \mathcal{M}$), $t_1 = \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\} = \min \emptyset = \infty$ and thus $\mathbb{T} = (\infty)$. On the other hand, since $\bar{\mathcal{M}} \setminus \mathcal{M} = \mathcal{F}_R$ the minimum in (6) exists for any $\mathbf{x}_0 \in \mathbb{R}^n \setminus \mathcal{M}$ if and only if it exists for any $\mathbf{x}_0 \in \mathcal{F}_R \setminus \{\mathbf{0}\}$. In addition, $e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}$ for some $\Delta \in \mathbb{R}^+$ if and only if $e^{A\Delta}(\alpha\mathbf{x}_0) \in \mathcal{M}$ for any $\alpha \in \mathbb{R} \setminus \{0\}$. As a result, $\min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\}$ exists for any $\mathbf{x}_0 \in \mathbb{R}^n \setminus \mathcal{M}$ if and only if the minimum exists for any $\mathbf{x}_0 \in \mathcal{HS}^{n-1} \cap \mathcal{F}_R$, where \mathcal{HS}^{n-1} is the unit $(n-1)$ -hemisphere in \mathbb{R}^n centered at the origin $\mathbf{0}$. Therefore, it is true that $t_1 \geq 0$, in fact $t_1 = 0$ if $\mathbf{x}_0 \in \mathcal{M}$, and $t_1 > 0$ otherwise.

For the following reset instants the reasoning is similar. The first after-reset state is $\mathbf{x}_1 = A_R e^{At_1}\mathbf{x}_0 \notin \mathcal{M}$ (note that $\mathbf{x}_1 \in \mathcal{M}_R$), and the second reset instant t_2 is given by $t_2 = t_1 + \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_1 \in \mathcal{M}\}$, for some $\mathbf{x}_1 \in \mathcal{M}_R$. For $\mathbf{x}_1 = \mathbf{0}$, $t_2 = \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_1 \in \mathcal{M}\} = \min \emptyset = \infty$, and for $\mathbf{x}_1 \in \mathcal{M}_R \setminus \{\mathbf{0}\}$ the minimum exists if and only if it exist for $\mathbf{x}_1 \in \mathcal{HS}^{n-1} \cap \mathcal{F}_R$ by using the above argument. The same reasoning is again applied for the rest of the reset instants. \square

This Proposition reduces the dimensionality when checking whether a reset system has well-posed reset instants or not, simply by checking if a minimum exists for a reduced number of states that are elements of the hemisphere $\mathcal{HS}^{n-1} \cap \mathcal{F}_R$. This is particularly simple for low-order reset systems, as shown in the next

examples. Moreover, for reset systems with well-posed reset instants, a function $\tau_{\mathcal{HS}} : \mathcal{HS}^{n-1} \cap \bar{\mathcal{M}}_R \rightarrow \mathbb{R}^+$ is defined (note that $\bar{\mathcal{M}}_R = \mathcal{M}_R \cup \mathcal{F}_R$):

$$\tau_{\mathcal{HS}}(\mathbf{x}) := \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x} \in \mathcal{M}\} \quad (7)$$

For simplicity of notation, it is also convenient to define a function $\tau_S : \mathcal{S}^{n-1} \cap \bar{\mathcal{M}}_R \rightarrow \mathbb{R}^+$ such that

$$\tau_S(\mathbf{x}) := \begin{cases} \tau_{\mathcal{HS}}(\mathbf{x}) & , \mathbf{x} \in \mathcal{HS}^{n-1} \cap \bar{\mathcal{M}}_R \\ \tau_{\mathcal{HS}}(-\mathbf{x}) & , otherwise \end{cases} \quad (8)$$

Thus, given the first reset instant t_1 , the sequence $\mathbb{T} = (t_1, t_2, \dots)$ may be obtained as

$$\begin{cases} t_{k+1} = t_k + \tau_S\left(\frac{\mathbf{x}_k}{\|\mathbf{x}_k\|}\right) \\ \mathbf{x}_k = A_R e^{At_k} \mathbf{x}_{k-1} \end{cases} \quad (9)$$

for $k = 1, 2, \dots$. And the first reset instant t_1 is

$$t_1 = \begin{cases} 0 & , \mathbf{x}_0 \in \mathcal{M} \\ \tau_S\left(\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|}\right) & , \mathbf{x}_0 \in \bar{\mathcal{M}}_R \setminus \{\mathbf{0}\} \\ \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\} & , otherwise \end{cases} \quad (10)$$

In addition, (7-10) may be used to define functions $\tau_i : \mathbb{R}^n \rightarrow \mathbb{R}$, for $i = 1, 2, \dots, N$, such that $t_i = \tau_i(\mathbf{x}_0)$ is the i^{th} element of the sequence of reset instants \mathbb{T} . These functions take an important role in the impulsive systems literature ([5], [28], [36]).

Example III.3: For the reset system of example III.1, $\mathcal{HS}^{n-1} \cap \mathcal{F}_R = \emptyset$, and thus it has directly well-posed reset instants. In addition, $\mathcal{HS}^{n-1} \cap \bar{\mathcal{M}}_R = \{(1, 0)\}$ (the point A in Fig. 1a), and $\tau_{\mathcal{HS}}((1, 0)) = \frac{\pi}{4\omega}$, thus the reset instants may be easily computed since they are periodic with period $\frac{\pi}{4\omega}$ after the second reset instant. On the other hand, for the reset system of Example III.2, for $\mathbf{x}_0 \in \mathcal{HS}^{n-1} \cap \mathcal{F}_R = \{(0, 1, 0)\}$ the first reset instant $t_1 = \min(0, T_1)$ is not well defined since $(0, T_1)$ is open and thus the minimum does not exist; therefore, the reset system has not well-posed reset instants.

Example III.4: Consider a reset system $(A, C, 1)$ with

$$A = \begin{pmatrix} 0 & 0 & -0.35 & 3 \\ 1 & 0 & -2.40 & 1 \\ 0 & 1 & -4.35 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \quad (11)$$

In this case, the set $\mathcal{S}^{n-1} \cap \bar{\mathcal{M}}_R = \mathcal{S}^{n-1} \cap \mathcal{F}_R$ is the unit circumference centered at the origin of the plane $x_1 - x_2$. It may be parameterized by

$$\mathbf{x}(\theta) = (\cos(\theta), \sin(\theta), 0, 0) \quad (12)$$

for $\theta \in [0, 2\pi)$ and τ_S may be computed as a function of θ , that is $\tau_S(\theta) := \tau_S(\mathbf{x}(\theta))$ (see Fig. 3), by using (7)-(8) (note that $\tau_S(\theta + \pi) = \tau_S(\theta)$ for $\theta \in [0, \pi)$). This is equivalent to solve for t the implicit equation

$$C e^{At} (\cos(\theta), \sin(\theta), 0, 0) = 0 \quad (13)$$

The result is that the domain of the map $\tau_{\mathcal{HS}}$ is $\mathcal{HS}^{n-1} \cap \mathcal{F}_R$ (and thus the domain of τ_S is $\mathcal{S}^{n-1} \cap \mathcal{F}_R$), and then by Prop. III.1 the reset system has well-posed reset instants. On the other hand, τ_S has a discontinuity at $\theta = \pi$. As a result, functions τ_i , $i = 1, \dots, N$ are also discontinuous. It is worthwhile to mention that the continuity of these functions is a common assumption in most of the work done about IDS [28], and thus it is not directly applicable to reset systems. In the following, a geometric condition based on the system matrices A , A_R , and C will be developed for the reset system (A, C, n_r) to have well-posed reset instants.

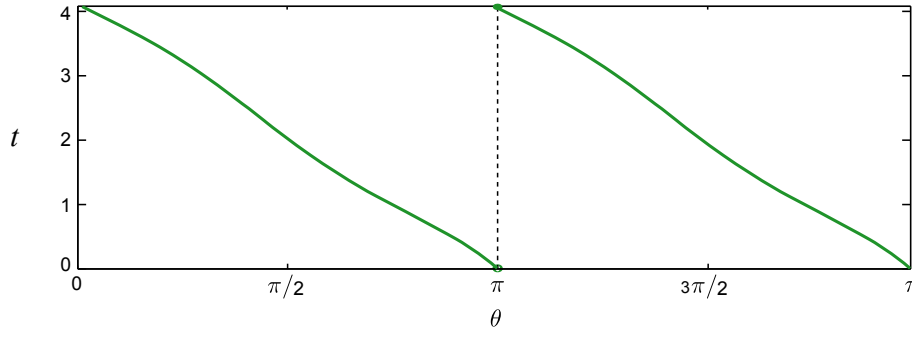


Fig. 5. Reset instants corresponding to states in $\mathcal{S}^{n-1} \cap \mathcal{M}_R$ as a function of the parameter θ , $t = \tau_S(\theta)$ (note that $\tau_S(\pi^-) = 0$, and $\tau_S(\pi) = \tau_S(\pi^+) \approx 4.13$).

Proposition III.2: *The reset system (A, C, n_r) has well-posed reset instants if and only if the subspace*

$$\mathcal{F}_{RU} := \mathcal{N}\left(\begin{pmatrix} I - A_R \\ \mathcal{O}_{base} \end{pmatrix}\right) \quad (14)$$

is A -invariant, where \mathcal{O}_{base} is the observability matrix of the base system.

Proof: First note that $\mathcal{F}_{RU} = \mathcal{F}_R \cap \mathcal{N}(\mathcal{O}_{base})$ and thus \mathcal{F}_{RU} is the subspace of fixed points of A_R that are unobservable; on the other hand, for a given $\mathbf{x}_0 \in \mathbb{R}^n$, the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(\Delta) = Ce^{A\Delta}\mathbf{x}_0$ is an analytical function (it is a Bohl function). A property of f to be used below is that either $f(\Delta) = 0$ for any $\Delta \in \mathbb{R}$ or f has isolated zeros.

By Prop. III.1, that (A, C, n_r) has well-posed reset instants is equivalent to the existence of $\min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\}$ for any $\mathbf{x}_0 \in \mathcal{HS}^{n-1} \cap \mathcal{F}_R$, or equivalently for any $\mathbf{x}_0 \in \mathcal{F}_R$.

if) For $\mathbf{x}_0 \in \mathcal{F}_R$, either $f(\Delta) = Ce^{A\Delta}\mathbf{x}_0 = 0$ for any $\Delta \in \mathbb{R}$ or f has isolated zeros. If $f(\Delta) = 0$ for any $\Delta \in \mathbb{R}$, then $\mathbf{x}_0 \in \mathcal{F}_R \cap \mathcal{N}(\mathcal{O}_{base}) = \mathcal{F}_{RU}$, and since \mathcal{F}_{RU} is A -invariant then $e^{A\Delta}\mathbf{x}_0 \notin \mathcal{M}$. Thus, $\min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\} = \min \emptyset = +\infty$. On the other hand, if f has isolated zeros then $Ce^{A\Delta}\mathbf{x}_0 \neq 0$ and thus $e^{A\Delta}\mathbf{x}_0 \notin \mathcal{M}$ for $\Delta \in (0, \epsilon)$ and some constant $\epsilon > 0$. The result is that $\min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\}$ does also exist in this case.

only if) By contradiction, if $\mathbf{x}_0 \in \mathcal{F}_{RU}$ and \mathcal{F}_{RU} is not A -invariant then $f(\Delta) = Ce^{A\Delta}\mathbf{x}_0 = 0$ for any $\Delta \in \mathbb{R}$, and $e^{A\Delta}\mathbf{x}_0 \in \mathcal{N}(\mathcal{O}_{base}) \setminus \mathcal{F}_{RU} \subset \mathcal{M}$, for $\Delta \in (0, \epsilon)$ and some constant $\epsilon > 0$. Thus $\min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \mathcal{M}\}$ does not exist for $\mathbf{x}_0 \in \mathcal{F}_{RU}$, which is a contradiction. \square

Example III.5: For the reset system of Example III.1, the after reset set is $\mathcal{F}_R = \{0\}$. In this case, $\mathcal{F}_{RU} = \{0\}$ is trivially A -invariant, and thus the reset system has well-posed reset instants according to Prop. III.2. In the case of Example III.2, the after reset set \mathcal{F}_R is the x_2 -axis and the unobservable subspace of the base system is $\mathcal{N}(\mathcal{O}_{base}) = \text{span}\{(0, 0, 1), (0, 1, 0)\}$ (the $x_2 - x_3$ plane); in addition, \mathcal{F}_{RU} is the x_2 -axis and $A(\mathcal{F}_{RU}) = \text{span}\{(0, -1, 1)\} \not\subset \mathcal{F}_{RU}$; as a result, \mathcal{F}_{RU} is not A -invariant and thus the reset system has ill-posed reset instants.

By definition, a *Zeno solution* \mathbf{x} of the IDS (1) exists for some initial condition $\mathbf{x}_0 \in \mathbb{R}^n$ if there exists an infinite sequence of reset instants $\mathbb{T} = (t_k)_{k=1}^\infty$, and $t_\infty \in \mathbb{R}^+$ such as $t_k \rightarrow t_\infty$ as $k \rightarrow \infty$. Note that for a reset system with well-posed reset instants, a solution \mathbf{x} exists on $I_{\mathbf{x}_0}$ for any $\mathbf{x}_0 \in \mathbb{R}^n$, where $I_{\mathbf{x}_0} = [0, t_\infty)$ if it is a Zeno solution, and $I_{\mathbf{x}_0} = [0, \infty)$ otherwise. A simple counterexample ([9], [18]) shows that in general Zeno solutions may exist for reset systems with well-posed reset instants.

B. Reset control systems

In this work, a reset control system (Fig. 6) refers to a feedback interconnection of a LTI system P and a reset compensator R with base system R_{base} . P is described by:

$$\begin{cases} \dot{\mathbf{x}}_p(t) = A_p \mathbf{x}_p(t) + B_p u(t) \\ y(t) = C_p \mathbf{x}_p(t) \end{cases} \quad (15)$$

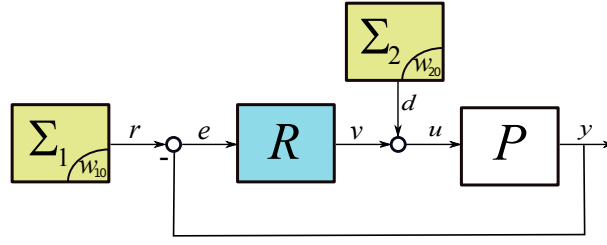


Fig. 6. Reset control system (exosystems have not inputs, their initial conditions are explicitly shown in the bottom right corner).

with $\mathbf{x}_p(0) = \mathbf{x}_{p0}$, and the reset compensator R is given by

$$\begin{cases} \dot{\mathbf{x}}_r(t) = A_r \mathbf{x}_r(t) + B_r e(t), & e(t) \neq 0 \vee (I - A_\rho) \mathbf{x}_r(t) = \mathbf{0} \\ \mathbf{x}_r(t^+) = A_\rho \mathbf{x}_r(t), & e(t) = 0 \wedge (I - A_\rho) \mathbf{x}_r(t) \neq \mathbf{0} \\ v(t) = C_r \mathbf{x}_r(t) + D_r e(t) \end{cases} \quad (16)$$

with $\mathbf{x}_r(0) = \mathbf{x}_{r0}$. Here $\mathbf{x}_p \in \mathbb{R}^{n_p}$, $\mathbf{x}_r \in \mathbb{R}^{n_r}$, and $e, u \in \mathbb{R}$. It is assumed that the last n_ρ compensator states are set to zero at the reset instants, then A_ρ is partitioned in blocks as

$$A_\rho = \begin{pmatrix} I_{n_{\bar{\rho}}} & O_{n_{\bar{\rho}} \times n_\rho} \\ O_{n_\rho \times n_{\bar{\rho}}} & O_{n_\rho \times n_\rho} \end{pmatrix} \quad (17)$$

where $n_{\bar{\rho}} = n_r - n_\rho$, and in addition A_r , B_r , and C_r are partitioned into blocks with appropriate block dimensions:

$$A_r = \begin{pmatrix} A_{r11} & A_{r12} \\ A_{r21} & A_{r22} \end{pmatrix}, B_r = \begin{pmatrix} B_{r1} \\ B_{r2} \end{pmatrix}, C_r = (C_{r1} \ C_{r2}) \quad (18)$$

In the case of a *full reset* compensator, all the elements of A_ρ are 0; otherwise, R is a *partial reset* compensator.

As it is usual in control practice, reset control systems are driven by external or exogenous inputs such as reference or disturbance signals (note that output measurement noise may be included in the reference signal for analysis of existence and uniqueness of solutions). It will be assumed that the reference input r and the disturbance input d are generated by exosystems Σ_1 and Σ_2 respectively, with state space models

$$\Sigma_1 : \begin{cases} \dot{\mathbf{w}}_1(t) &= A_1 \mathbf{w}_1(t), & \mathbf{w}_1(0) = \mathbf{w}_{10} \\ r(t) &= C_1 \mathbf{w}_1(t), \end{cases} \quad (19)$$

with $\mathbf{w}_1 \in \mathbb{R}^{m_1}$, and

$$\Sigma_2 : \begin{cases} \dot{\mathbf{w}}_2(t) &= A_2 \mathbf{w}_2(t), & \mathbf{w}_2(0) = \mathbf{w}_{20} \\ d(t) &= C_2 \mathbf{w}_2(t), \end{cases} \quad (20)$$

with $\mathbf{w}_2 \in \mathbb{R}^{m_2}$. These exosystems allow to generate signals like steps, ramps, sinusoids, etc. (Bohl functions). Now, the feedback connection is obtained by making $e = r - y$ and $u = v + d$. Define the closed-loop state as $\mathbf{x} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{x}_p, \mathbf{x}_r)$, then the reset control system of Fig. 4 can be represented as a reset system (A, C, n_ρ) with

$$A = \begin{pmatrix} A_1 & O & O & O \\ O & A_2 & O & O \\ B_p D_r C_1 & B_p C_2 & A_p - B_p D_r C_p & B_p C_r \\ B_r C_1 & O & -B_r C_p & A_r \end{pmatrix} \quad (21)$$

$$C = (C_1 \ O \ -C_p \ O)$$

Note that, according to Def. II.1, the reset set is $\mathcal{M} = \mathcal{N}(C) \setminus \mathcal{F}_R$, and thus R performs reset actions at the instant t when $e(t) = C\mathbf{x}(t) = 0$ only if $\mathbf{x}(t) \notin \mathcal{F}_R$, that is when $e(t) = 0$ and $(I - A_\rho)\mathbf{x}_r(t) \neq \mathbf{0}$. In

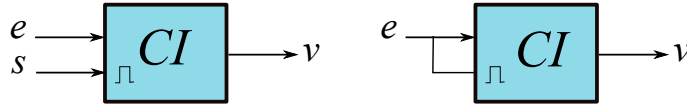


Fig. 7. (left) Two-inputs Clegg integrator, (right) Clegg integrator (by simplicity, the connection of the signal e to the trigger input means that s is the boolean expression $e = 0 \wedge v \neq 0$).

addition, since the last n_ρ values of C are zero, then $\mathcal{M}_R = \mathcal{F}_R$, and therefore $\mathcal{M}_{RU} := \mathcal{M}_R \cap \mathcal{N}(\mathcal{O}_{base}) = \mathcal{F}_{RU}$.

For a block diagram representation of the reset compensator R as given by (16)-(18), it is sufficient to employ an extension of the Clegg integrator as shown in Fig. 7, in which $s : \mathbb{R}^+ \rightarrow \{0, 1\}$ is a boolean-valued trigger function that takes values 0 (false) and 1 (true):

$$CI : \begin{cases} \dot{v}(t) = e(t), & s(t) = 0 \\ v(t^+) = 0, & s(t) = 1 \end{cases} \quad (22)$$

A block diagram of R that allows a direct practical implementation is given in Fig. 8. On the other hand, if $A_{r_{21}} = O$ ($A_{r_{12}} = O$) then R will be referred to as a *right reset compensator* (*left reset compensator*); the name is related with the right (left) triangular block structure of the matrix A_r . It is worthwhile to mention that some of the reset compensator with partial reset (see Fig. 9) that has been found useful in practice [9] are right reset compensators.

In the following, a necessary and sufficient condition for existence and uniqueness of solutions will be developed; and, in addition, this result will be applied to reset control systems with a full/partial reset compensation structure.

Proposition III.3: *The reset control system (A, C, n_ρ) , with A and C given by (21), has a unique solution on $[0, \infty)$, for any $\mathbf{x}_0 \in \mathbb{R}^n$, if and only if it has well-posed reset instants.*

Proof: *only if*) By contradiction, if (A, C, n_ρ) does not have well-posed reset instants, then for some initial condition $\mathbf{x}_0 \in \mathbb{R}^n$ the reset instant sequence \mathbb{T} is not well-defined, that is for some integer k , $1 \leq k < \infty$, $\min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}x_{k-1} \in \mathcal{M}\}$ does not exist and thus the solution is not defined for $t > t_{k-1}$ and some finite t_{k-1} . This is in contradiction with the solution to be defined on $[0, \infty)$.

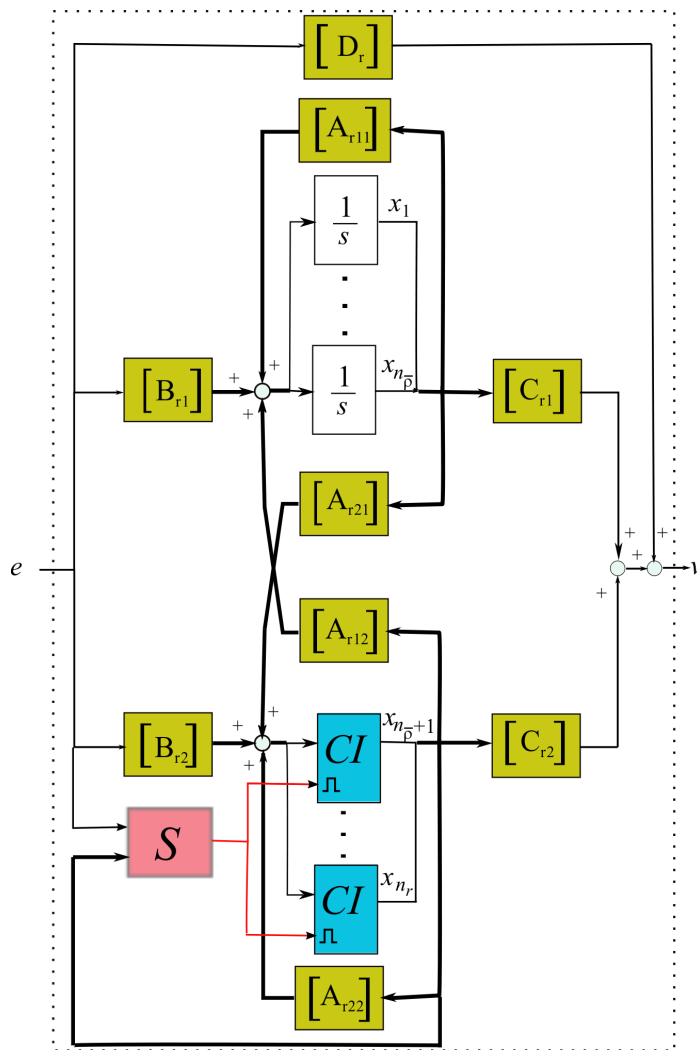
if) By well-posedness of the reset instants, \mathbb{T} is well-defined for any $\mathbf{x}_0 \in \mathbb{R}^n$. If the sequence \mathbb{T} is finite then the result directly follows by existence and uniqueness of solutions of the LTI base system; otherwise, it will be shown that if $\mathbb{T} = (t_k)_{k=1}^\infty$ then $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Since for $k \geq 2$ it is true that $\mathbf{x}(t_k^+) \in \mathcal{M}_R$, to complete the proof it is enough with showing that $t_k \rightarrow \infty$ as $k \rightarrow \infty$ for any $\mathbf{x}_0 \in \mathcal{M}_R$ with an infinite sequence \mathbb{T} . This directly follows from the following result: an initial condition in the after-reset set \mathcal{M}_R , with dimension m , will have sequences of decreasing reset intervals with length at most $m - 1$ (a detailed proof is given in [8], [9], note that $\mathcal{F}_R = \mathcal{M}_R$). Thus, since for the base system, solutions exists and are unique for any initial condition, it directly follows that the reset system is well-posed. \square

Since the only way in which Zeno solutions may exist is that reset instants be well-posed, from Prop. III.3 it may be concluded that reset control systems do not have Zeno solutions, since the solution is defined on $[0, \infty)$ when reset instants are well-posed (note that this is only true for reset control systems in which P has a strictly proper transfer function as given by (15)). Thus:

- Ill-posed reset instants implies the existence of deadlock for some initial condition, but not the existence of Zeno solutions.
- Well-posed reset instants implies that neither deadlock nor Zeno solutions do exist.

C. Full reset and right reset compensation

A natural question to ask is whether existence and uniqueness of reset control system solutions can be checked in a simple manner for a given system P and a reset compensator R , and for any exogenous



inputs, that is with independence of the exosystems. By using Prop. III.2 and III.3, this is about to derive simple conditions for the subspace $\mathcal{M}_{RU} = \mathcal{N}(I - A_R) \cap \mathcal{N}(\mathcal{O}_{base})$ to be A -invariant.

Note that if the base system is observable, that is \mathcal{O}_{base} is full rank, then $\mathcal{M}_{RU} = \{\mathbf{0}\}$ and the condition is trivially satisfied; but this is also the case if $\dim(\mathcal{N}(\mathcal{O}_{base})) = 1$, that is if there exists only one unobservable mode, since $\dim(\mathcal{M}_{RU}) \leq \dim(\mathcal{N}(\mathcal{O}_{base})) = 1$ then $\mathcal{M}_{RU} = \{\mathbf{0}\}$ or $\mathcal{M}_{RU} = \mathcal{N}(\mathcal{O}_{base})$ (in both cases \mathcal{M}_{RU} is A -invariant). The general case is much more involved; in the following, the cases of full reset and right reset compensation are analyzed.

Proposition III.4: *The reset control system (A, C, n_ρ) , with A and C given by (21), has well-posed reset instants if the reset compensator is full reset or partial reset with right reset.*

Proof: Regroup states as $\bar{\mathbf{x}}_p = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{x}_p)$, and split the compensator state \mathbf{x}_r into two parts, $\mathbf{x}_r = (\mathbf{x}_{\bar{p}}, \mathbf{x}_\rho)$, where $\mathbf{x}_{\bar{p}}$ and \mathbf{x}_ρ , corresponding to non-resetting and resetting states respectively. Thus the closed-loop state is $\mathbf{x} = (\bar{\mathbf{x}}_p, \mathbf{x}_{\bar{p}}, \mathbf{x}_\rho)$. Moreover, by using submatrices with appropriate dimensions, matrices A ,

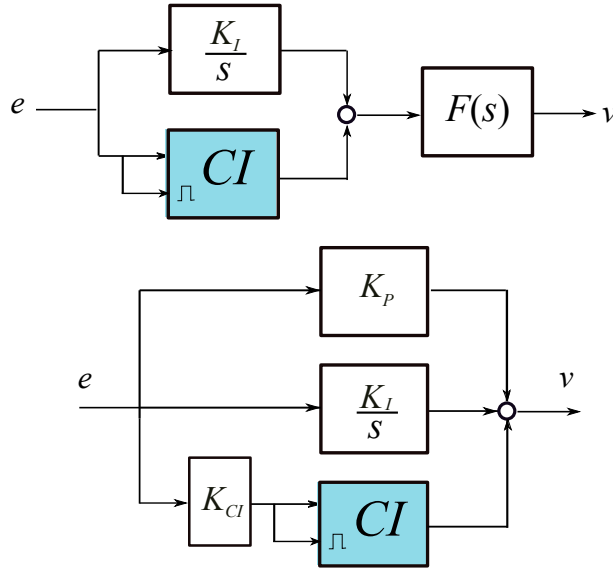


Fig. 9. Right-reset compensators: (top) Horowitz reset compensator ([31], K_I is a parameter to be tuned and $F(s)$ a transfer function to be designed), (bottom) PI+CI compensator ([9], [47], K_P , K_I and K_{CI} are compensator parameters to be tuned).

A_R and C are partitioned as

$$A = \begin{pmatrix} \bar{A}_p & \bar{B}_p C_{r1} & \bar{B}_p C_{r2} \\ -B_{r1} \bar{C}_p & A_{r11} & A_{r12} \\ B_{r2} \bar{C}_p & A_{r21} & A_{r22} \end{pmatrix}, \quad A_R = \begin{pmatrix} I & O & O \\ O & I & O \\ O & O & O \end{pmatrix} \quad (23)$$

$$C = (\bar{C}_p \quad O \quad O)$$

Now, using (3) the subspace of after-reset and unobservable states $\mathcal{M}_{RU} = \mathcal{F}_{RU} = \mathcal{N}(\begin{pmatrix} I - A_R \\ \mathcal{O}_{base} \end{pmatrix})$. For any state $\mathbf{x} \in \mathcal{M}_{RU}$, it is true that $\mathbf{x}_\rho = \mathbf{0}$, and $\bar{C}_p \bar{\mathbf{x}}_p = C\mathbf{x} = 0$, and then

$$(I - A_R)A\mathbf{x} =$$

$$\begin{pmatrix} O & O & O \\ O & O & O \\ O & O & I \end{pmatrix} \begin{pmatrix} \bar{A}_p & \bar{B}_p C_{r1} & \bar{B}_p C_{r2} \\ -B_{r1} \bar{C}_p & A_{r11} & A_{r12} \\ B_{r2} \bar{C}_p & A_{r21} & A_{r22} \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}}_p \\ \mathbf{x}_{\bar{\rho}} \\ \mathbf{0} \end{pmatrix} =$$

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ B_{r2} \bar{C}_p \bar{\mathbf{x}}_p + A_{r21} \mathbf{x}_{\bar{\rho}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ A_{r21} \mathbf{x}_{\bar{\rho}} \end{pmatrix} \quad (24)$$

As a result, \mathcal{M}_{RU} is A -invariant if and only if $A_{r21} \mathbf{x}_{\bar{\rho}} = \mathbf{0}$ for any $(\bar{\mathbf{x}}_p \quad \mathbf{x}_{\bar{\rho}} \quad \mathbf{0}) \in \mathcal{N}(\mathcal{O}_{base})$. The result follows since for a full reset compensator $\mathbf{x}_{\bar{\rho}} = \mathbf{0}$, and for a right reset compensator $A_{r21} = 0$. \square

Example III.6 (full reset compensation with ill-posed reset instants): Example III.2 describes a full reset system with ill-posed reset instants; note that the reset system does not correspond to a reset control system as given in Fig. 6.

Example III.7 (reset control system with reference input and well-posed reset instants): Consider the reset control system of Fig. 6, where R is a CI and P is an integrator. In addition, consider a sinusoidal reference $r(t) = A \sin(\omega t + \phi)$, for some given constants A , $\omega > 0$, and ϕ ; it is given by the exosystem

$$\begin{cases} \dot{\mathbf{w}}_1(t) = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \mathbf{w}_1(t), & \mathbf{w}_1(0) = \begin{pmatrix} A \sin \phi \\ A \cos \phi \end{pmatrix} \\ r(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{w}_1 \end{cases} \quad (25)$$

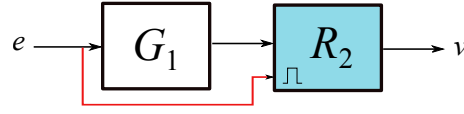


Fig. 10. A left reset compensator

Since a disturbance signal is not considered in this example, Prop. III.3 can be used by simply eliminating the row and column blocks corresponding to the disturbance exosystem in the matrices A , A_R , and C . The result is

$$A = \begin{pmatrix} 0 & \omega & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix} \quad (26)$$

Now, the observability matrix of the (closed-loop) base system is

$$O_{base} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & \omega & 0 & -1 \\ -(1 + \omega^2) & 0 & 1 & 0 \\ 0 & -\omega(1 + \omega^2) & 0 & 1 \end{pmatrix} \quad (27)$$

which is full rank for any $\omega > 0$. As a result $\mathcal{M}_{RU} = \{\mathbf{0}\}$ is trivially A -invariant and thus the system is well-posed for arbitrary sinusoidal reference inputs. Note that, since Proposition III.4 applies (the reset compensator R is a Clegg integrator and thus it is full reset), in this case it is not necessary to check the A -invariance of the subspace \mathcal{M}_{RU} ; and moreover, it is possible to assure a much more general result: the reset control system has well-posed reset instants not only for sinusoidal references but for any exogenous inputs generated by exosystems.

D. Partial reset compensation (left reset compensators)

The general case of partial reset compensation is much more involved; in the following, existence and uniqueness of solutions will be analyzed for a type of left reset compensators, in particular for reset compensators R that are a series interconnection of a LTI compensator $G_1 = (A_{r1}, B_{r1}, C_{r1})$ with state $\mathbf{x}_{\bar{p}} \in \mathbb{R}^{n_{\bar{p}}}$, and a full reset compensator R_2 with a base system $R_{2,b} = (A_{r2}, B_{r2}, C_{r2})$, and with state $\mathbf{x}_{\rho} \in \mathbb{R}^{n_{\rho}}$ (Fig. 10).

Thus, R has a base system $R_b = (A_r, B_r, C_r)$ with

$$A_r = \begin{pmatrix} A_{r1} & O \\ B_{r2}C_{r1} & A_{r2} \end{pmatrix} \quad B_r = \begin{pmatrix} B_{r1} \\ O \end{pmatrix} \quad (28)$$

$$C_r = \begin{pmatrix} O & C_{r2} \end{pmatrix}$$

and thus it is a left reset compensator. Now, consider R as the compensator of the reset control system (A, C, n_{ρ}) of Fig. 7, with A and C given by (23). In addition, by defining again the closed loop state as $\mathbf{x} = (\bar{\mathbf{x}}_p, \mathbf{x}_{\bar{p}}, \mathbf{x}_{\rho})$, with $\bar{\mathbf{x}}_p = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{x}_p)$, A and C can be partitioned as:

$$A = \begin{pmatrix} \bar{A}_p & O & \bar{B}_p C_{r2} \\ -B_{r1} \bar{C}_p & A_{r1} & O \\ O & B_{r2} C_{r1} & A_{r2} \end{pmatrix} \quad C = \begin{pmatrix} \bar{C}_p & O & O \end{pmatrix} \quad (29)$$

It will be assumed that the realizations (A_r, B_r, C_r) and (A_p, B_p, C_p) are minimal, and that $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$ is observable¹. Thus, since in this case of left reset compensation the base control system of Fig. 7 is

¹Using interconnection properties, note that $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$ is simply a parallel connection of a system (A_1, O, C_1) and the series connection of (A_2, O, C_2) and (A_p, B_p, C_p) (see Fig. 7), thus it directly follows that unobservable modes of $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$ are given by common eigenvalues of A_1 and A_2 , common values of eigenvalues of A_1 and modes of (A_p, B_p, C_p) , and common values of eigenvalues of A_2 and zeros of (A_p, B_p, C_p) .

simply a feedback connection of the series connection of $(A_{r_1}, B_{r_1}, C_{r_1})$, $(A_{r_2}, B_{r_2}, C_{r_2})$, and $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$, it is clear that every unobservable mode $\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)$ of the base control system must be a pole of $(A_{r_1}, B_{r_1}, C_{r_1})$ and/or $(A_{r_2}, B_{r_2}, C_{r_2})$, and a zero of $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$.

For an unobservable mode λ , let m_λ be the algebraic multiplicity as a zero of $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$; q_λ , the pole algebraic multiplicity in $(A_{r_1}, B_{r_1}, C_{r_1})$; and r_λ , the pole algebraic multiplicity in $(A_{r_2}, B_{r_2}, C_{r_2})$. In addition, the number of cancellations is $d_\lambda = \min(q_\lambda + r_\lambda, m_\lambda)$. On the other hand, since the geometric multiplicity of the unobservable modes in the base control system is 1 (see Prop. A.1 in the Appendix), then the index of an unobservable mode of A is equal to its algebraic multiplicity ma_λ , that in general will be greater or equal than d_λ . In addition, the dimension of $\mathcal{N}(\mathcal{O})$ is the sum of all the cancellations corresponding to the unobservables modes (see Proposition A.3 in the Appendix), that is

$$\dim(\mathcal{N}(\mathcal{O})) = s := \sum_{\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)} d_\lambda \quad (30)$$

Proposition III.5: Consider the reset control system (A, C, n_ρ) of Fig. 7, with P given by (17) and R being a left reset compensator with base system $R_b = (A_r, B_r, C_r)$ as given by (32). (A, C, n_ρ) has well-posed reset instants if and only if

$$n_\rho \geq s \quad (31)$$

Proof: (if): By using Prop. A.2 and Prop. III.2, it is sufficient to prove that $\mathcal{M}_{RU} = \mathcal{M}_R \cap \mathcal{N}(\mathcal{O}) = \{0\}$. In virtue of Proposition A.3, let $\{\mathbf{v}_\lambda^{(d_\lambda-k)} \in \mathbb{R}^n : \lambda \in \sigma_{\bar{\mathcal{O}}}(A, C), k = 0, \dots, d_\lambda - 1\}$ be a basis of $\mathcal{N}(\mathcal{O})$ with $\mathbf{v}_\lambda^{(d_\lambda-k)} = (\mathbf{v}_{p_\lambda}^{(d_\lambda-k)}, \mathbf{v}_{\bar{\rho}_\lambda}^{(d_\lambda-k)}, \mathbf{v}_{\rho_\lambda}^{(d_\lambda-k)})$ being a generalized eigenvector of A corresponding to the unobservable mode λ ($\mathbf{v}_\lambda^{(d_\lambda)}$ is the eigenvector). Thus, any $\mathbf{v} = (\mathbf{v}_p, \mathbf{v}_{\bar{\rho}}, \mathbf{v}_\rho) \in \mathcal{N}(\mathcal{O})$ can be expressed as $\mathbf{v} = \sum_{\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)} \sum_{k=0}^{d_\lambda-1} \alpha_\lambda^{(d_\lambda-k)} \mathbf{v}_\lambda^{(d_\lambda-k)}$, where $\mathbf{v}_\rho = \sum_{\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)} \sum_{k=0}^{d_\lambda-1} \alpha_\lambda^{(d_\lambda-k)} \mathbf{v}_{\rho_\lambda}^{(d_\lambda-k)}$. In the following, it will be shown that if $\mathbf{v} \in \mathcal{N}(\mathcal{O}) \cap \mathcal{M}_R$, and thus $\mathbf{v}_\rho = 0$, then the scalars $\alpha_\lambda^{(d_\lambda-k)}$ are all zero and thus $\mathbf{v} = 0$. Note that, in general $\{\mathbf{v}_{\rho_\lambda}^{(d_\lambda-k)} : \lambda \in \sigma_{\bar{\mathcal{O}}}(A, C), k = 0, \dots, d_\lambda - 1\}$ is not the set of generalized eigenvectors (including the eigenvector) of A_{r_2} corresponding to the mode λ , and thus it can not be directly concluded that it is a linearly independent set.

Consider a state transformation given by a matrix $S = \begin{pmatrix} I_{n_p} & O \\ O & T \end{pmatrix} \in \mathbb{R}^{n \times n}$ where $T = \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} M_1 & M_2 \\ N_1 & N_2 \end{pmatrix}$ with $M = \begin{pmatrix} M_1 & M_2 \end{pmatrix} \in \mathbb{R}^{n_{\bar{\rho}} \times (n_\rho + n_{\bar{\rho}})}$, and $N = \begin{pmatrix} N_1 & N_2 \end{pmatrix} \in \mathbb{R}^{n_\rho \times (n_\rho + n_{\bar{\rho}})}$. Following a procedure analogous to the Kalman decomposition (observable/unobservable decomposition), M can be selected in such a way that M_1 is nonsingular, $\mathcal{N}(M)$ is A_r -invariant, and that includes the subspace spanned by those generalized eigenvectors of A_r associated with unobservable modes of $(A, C, n_\rho)^2$. As a result if $T^{-1} = \begin{pmatrix} X & Y \end{pmatrix} = \begin{pmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{pmatrix}$ with $X \in \mathbb{R}^{(n_\rho + n_{\bar{\rho}}) \times n_{\bar{\rho}}}$ and $Y \in \mathbb{R}^{(n_\rho + n_{\bar{\rho}}) \times n_\rho}$, the matrix of the system is transformed into

$$\tilde{A} = SAS^{-1} = \begin{pmatrix} \bar{A}_p & \bar{B}_p C_{r_2}^T X_2 & \bar{B}_p C_{r_2}^T Y_2 \\ M B_r \bar{C}_p & \tilde{A}_{r_{11}} & O \\ N B_r \bar{C}_p & \tilde{A}_{r_{21}} & \tilde{A}_{r_{22}} \end{pmatrix}$$

It is straightforward to check that the new unobservable subspace $\mathcal{N}(\tilde{\mathcal{O}})$ with $\tilde{\mathcal{O}} = \mathcal{O}S^{-1}$ is spanned by a set $\{\tilde{\mathbf{v}}_\lambda^{(d_\lambda-k)} = (\mathbf{v}_{p_\lambda}^{(d_\lambda-k)}, 0, \tilde{\mathbf{v}}_{\rho_\lambda}^{(d_\lambda-k)}) : \lambda \in \sigma_{\bar{\mathcal{O}}}(A, C), k = 0, \dots, d_\lambda - 1\}$ where $\{\tilde{\mathbf{v}}_{\rho_\lambda}^{(d_\lambda-k)} \in \mathbb{R}_{\rho}^n : \lambda \in \sigma_{\bar{\mathcal{O}}}(A, C), k = 0, \dots, d_\lambda - 1\}$ is now a subset of the union of generalized eigenvectors sets of $\tilde{A}_{r_{22}}$

²The condition of being $\mathcal{N}(\mathcal{O})$ a subspace of the A_r -invariant subspace $\mathcal{N}(M)$ can be achieved by putting the basis of $\mathcal{N}(\mathcal{O})$ (composed by generalized eigenvectors of A) as rows of N , and then completing N with other generalized eigenvectors so as to obtain complete Jordan subchains of vectors (which spans cyclic subspaces). There exists freedom in order to select M (completion of a basis of row vectors) but always M_1 can be assured to be nonsingular by making M_1 to be a large multiple of the identity.

corresponding to the unobservable modes. Note that it is sufficient with the condition (35) for building such a subset.

Owing to $\mathcal{N}(\tilde{\mathcal{O}})$ is mapped into $\mathcal{N}(\mathcal{O})$ via S^{-1} it is clear that any $\mathbf{v} \in \mathcal{N}(\mathcal{O})$ can be written as $\mathbf{v} = \sum_{\lambda \in \sigma_{\tilde{\mathcal{O}}}(A,C)} \sum_{k=1}^{d_\lambda-1} \alpha_\lambda^{(d_\lambda-k)} S^{-1} \tilde{\mathbf{v}}_\lambda^{(d_\lambda-k)}$, for some scalars $\alpha_\lambda \in \mathbb{R}$.

Note that since $S^{-1} \tilde{\mathbf{v}}_\lambda^{(d_\lambda-k)} = \left(\mathbf{v}_{p_\lambda}^{(d_\lambda-k)}, Y_1 \tilde{\mathbf{v}}_{\rho_\lambda}^{(d_\lambda-k)}, Y_2 \tilde{\mathbf{v}}_{\rho_\lambda}^{(d_\lambda-k)} \right)$, it directly follows that $\mathbf{v} \in \mathcal{M}_R \cap \mathcal{N}(\mathcal{O})$ if and only if $\mathbf{0} = (I - A_R) \mathbf{v} = Y_2 \sum_{\lambda \in \sigma_{\tilde{\mathcal{O}}}(A,C)} \sum_{k=1}^{d_\lambda-1} \alpha_\lambda^{(d_\lambda-k)} \tilde{\mathbf{v}}_{\rho_\lambda}^{(d_\lambda-k)}$. In addition, since M_1 is not singular then

$$MY = M_1 Y_1 + M_2 Y_2 = O \Rightarrow Y_1 = -M_1^{-1} M_2 Y_2$$

$$NY = N_1 Y_1 + N_2 Y_2 = I \Rightarrow (-N_1 M_1^{-1} M_2 + N_2) Y_2 = I$$

and thus Y_2 is not singular with inverse $Y_2^{-1} = -N_1 M_1^{-1} M_2 + N_2$. Since in addition the set of generalized eigenvectors $\tilde{\mathbf{v}}_{\rho_\lambda}^{(d_\lambda-k)}$ is linearly independent then $\alpha_\lambda^{(d_\lambda-k)} = 0$ for all $\lambda \in \sigma_{\tilde{\mathcal{O}}}(A, C)$ and $k = 0, \dots, d_\lambda - 1$, and thus $\mathcal{M}_R \cap \mathcal{N}(\mathcal{O}) = \{\mathbf{0}\}$.

(only if): Let $\{\mathbf{v}_1, \dots, \mathbf{v}_s\}$ be a basis of $\mathcal{N}(\mathcal{O})$, where \mathbf{v}_i is partitioned as $\mathbf{v}_i = (\mathbf{v}_p^{(i)}, \mathbf{v}_{\bar{p}}^{(i)}, \mathbf{v}_\rho^{(i)})$, for $i = 1, \dots, s$. If (31) is false then $n_\rho < s$, and thus $\{\mathbf{v}_\rho^{(1)}, \dots, \mathbf{v}_\rho^{(s)}\}$ is a linearly dependent set. As a result, it is obtained that $\mathbf{0} = \sum_{i=1}^s \alpha_i \mathbf{v}_\rho^{(i)}$ for some scalars α_i , $i = 1, \dots, s$, not all zero. Now, using those

scalars the vector $\mathbf{v} = \sum_{i=1}^s \alpha_i \mathbf{v}_i$ must be nonzero since $\sum_{i=1}^s \alpha_i \mathbf{v}_i = \mathbf{0}$ if and only $\alpha_i = 0$ for $i = 1, \dots, s$.

As a result, there exists a nonzero $\mathbf{v} = (\mathbf{v}_p, \mathbf{v}_{\bar{p}}, \mathbf{v}_\rho) \in \mathcal{N}(\mathcal{O})$ with $\mathbf{v}_\rho = \mathbf{0}$, that is a nonzero $\mathbf{v} \in \mathcal{M}_{RU}$, and then by Proposition A.2 (see Appendix) and Proposition III.2 it follows that (A, C, n_ρ) has ill-posed reset instants. \square

Example III.8 (ill-posed reset control system with left reset compensation and disturbance input): Consider the reset control system of first row in Table I, with a sinusoidal disturbance input $d(t) = \sin(t)$ generated by a exosystem $\Sigma_2(s) = 1/(s^2 + 1)$, a system P with a transfer function $P(s) = 1/(s + 1)$, and a left reset compensator R given by the tandem connection of a (two inputs) Clegg integrator ($n_\rho = 1$), and a system G_1 with transfer function $G_1(s) = 1/(s^2 + 1)$. Note that in this case the unobservable modes are the common poles of $G_1(s)$ and $\Sigma_2(s)$, that is $\sigma_{\tilde{\mathcal{O}}} = \{+j, -j\}$, and in addition $q_{+j} = 1, r_{+j} = 0, m_{+j} = 1$, and $q_{-j} = 1, r_{-j} = 0, m_{-j} = 1$, and therefore $d_{+j} = 1$, and $d_{-j} = 1$. Since $n_\rho = 1 < d_{+j} + d_{-j} = 2$ then the reset control system is ill-posed.

This can be alternatively done, with some effort, by directly checking if \mathcal{M}_{RU} is A -invariant. The systems P , R , and the exosystem Σ_2 have the following realizations:

$$P : \quad A_p = -1 \quad B_p = 1 \quad C_p = 1 \quad (32)$$

$$R : \quad A_r = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad B_r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad C_r = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \quad (33)$$

$$\Sigma_2 : \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (34)$$

The reset control system $(A, C, 1)$, with state $\mathbf{x} = (\mathbf{w}_2, x_p, \mathbf{x}_r)$, is given by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (35)$$

$$\mathcal{M}_{RU} = \mathcal{N} \left(\begin{array}{c} I - A_R \\ \mathcal{O}_{base} \end{array} \right) = span\{ (0, 1, 0, -1, 0, 0) \} \quad (36)$$

On the other hand, consider the reset control system of the second row in Table I; in this case, the reset control system is well-posed since the compensator is a right reset compensator. This can be also concluded by checking that \mathcal{M}_{RU} is A -invariant (note that in this case $\mathcal{M}_{RU} = \text{span}\{(0, -1, 0, 1, 0, 0), (-1, 0, 0, 0, 1, 0)\} \neq$

$\{0\}$). Besides the above reset control systems, Table I shows several examples of reset control systems with well/ill-posed reset instants, based on the direct application of Prop. III.4-III.5 (note that reset systems with ill-posed reset instants must have base systems with at least two unobservable modes).

IV. CONTINUOUS DEPENDENCE ON THE INITIAL CONDITION

Several convenient metrics has been successfully developed in the literature to represent distances between impulsive/hybrid systems solutions; for example, the Skorokhod distance [15], and the graphical distance in the HI framework [16], [27]. In the particular case of IDSs, it will be shown that a convenient metric for determining the continuous dependence on the initial condition is directly the Hausdorff distance between the set of points defining the IDS trajectories; this metric has been used in [3] for analysis of impulsive integro-differential equations, and more recently in [23] for the analysis of continuous dependence of solutions of differential equations with nonfixed moments of impulses on the initial condition. In the following, this approach will be followed to develop analogous results about reset control systems.

Given two nonempty subsets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$, the *Hausdorff distance* between them is

$$d_H(\mathcal{A}, \mathcal{B}) := \max\{h(\mathcal{A}, \mathcal{B}), h(\mathcal{B}, \mathcal{A})\} \quad (37)$$

where $h(\mathcal{A}, \mathcal{B}) := \sup\{\inf\{\|a - b\|, a \in \mathcal{A}\}, b \in \mathcal{B}\}$. On the other hand, the euclidean distance $d_E(\mathcal{A}, \mathcal{B})$ is defined as

$$d_E(\mathcal{A}, \mathcal{B}) := \inf\{\inf\{\|a - b\|, b \in \mathcal{B}\}, a \in \mathcal{A}\} \quad (38)$$

For a function $\mathbf{f} : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ and some scalar $T > 0$, by definition $\mathbf{f}([0, T]) := \{\mathbf{f}(t) \in \mathbb{R}^n : t \in [0, T]\}$. In general, for two left-continuous functions \mathbf{f} and \mathbf{g} , a notion of distance using directly the Hausdorff distance may be problematical. Note that the distance $d_H(\mathbf{f}([0, T]), \mathbf{g}([0, T]))$ by itself does not give a good characterization of the property; for example, two functions like $\sin(t)$ and $\cos(t)$ would be at a distance 0 over the interval $[0, 2\pi]$, that is $d_H(\sin([0, 2\pi]), \cos([0, 2\pi])) = 0$ (similar examples can be easily found for left-continuous functions with jump discontinuities). In spite of this fact, it will be shown in next Section how Hausdorff distance can be successfully used. Note that for example $d_H(\sin([0, T]), \cos([0, T])) > 0$ if $T \in [0, \pi/2) \cup (\pi/2, 3\pi/2)$.

A. Definition and motivating examples

For a reset control system (A, C, n_ρ) , consider a solution \mathbf{x} corresponding to a initial condition \mathbf{x}_0 , and another solution \mathbf{x}^* corresponding to a perturbed initial condition \mathbf{x}_0^* . With some abuse of notation, let $\mathbf{x}([0, T], \mathbf{x}_0)$ be the set of points corresponding to the trajectory of the system for $t \in [0, T]$, that is

$$\mathbf{x}([0, T], \mathbf{x}_0) := \{\mathbf{x}(t) \in \mathbb{R}^n : t \in [0, T]\} \quad (39)$$

where \mathbf{x} is the solution of (A, C, n_ρ) with initial condition \mathbf{x}_0 . $\mathbf{x}([0, T])$ is used if the initial condition is clear from the context, and $\mathbf{x}^*([0, T], \mathbf{x}_0^*)$ is defined accordingly. Then, $d_H(\mathbf{x}([0, T], \mathbf{x}_0), \mathbf{x}^*([0, T], \mathbf{x}_0^*))$, or simply $d_H(\mathbf{x}([0, T]), \mathbf{x}^*([0, T]))$, will be referred to as the *Hausdorff distance* between the two trajectories.

Continuous dependence will be characterized by the property that for *almost all* $T \geq 0$, it is possible to find two trajectories $\mathbf{x}([0, T], \mathbf{x}_0)$ and $\mathbf{x}^*([0, T], \mathbf{x}_0^*)$ arbitrarily close, in the sense that $d_H(\mathbf{x}([0, T], \mathbf{x}_0), \mathbf{x}^*([0, T], \mathbf{x}_0^*))$ is arbitrarily small, by choosing close enough initial conditions \mathbf{x}_0 and \mathbf{x}_0^* .

Definition IV.1 (continuous dependence): For a reset control system (A, C, n_ρ) with well-posed reset instants, the solution depends continuously on the initial condition at $\mathbf{x}_0 \in \mathbb{R}^n$ if for any $\epsilon > 0$, $T \in \mathbb{R}^+ \setminus \mathbb{T}$, there exist $\delta > 0$ such that for any $\mathbf{x}_0^* \in \mathbb{R}^n$, with $\|\mathbf{x}_0 - \mathbf{x}_0^*\| < \delta$, it is true that

$$d_H(\mathbf{x}^*([0, T], \mathbf{x}_0^*), \mathbf{x}([0, T], \mathbf{x}_0)) < \epsilon \quad (40)$$

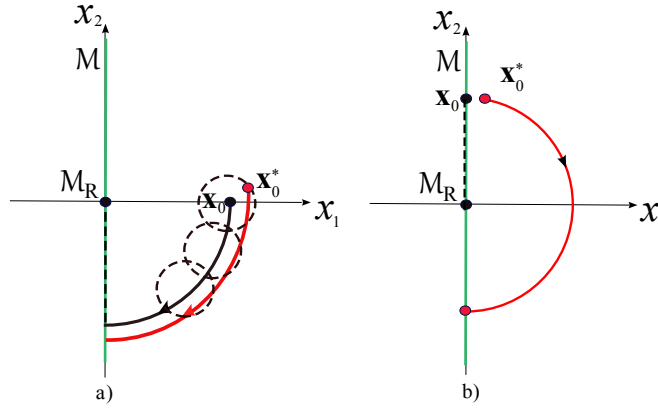


Fig. 11. \mathbf{x} (black) and \mathbf{x}^* (red): a) $\mathbf{x}_0 \in \mathbb{R}^n \setminus \mathcal{M}$, b) $\mathbf{x}_0 \in \mathcal{M}$.

A first analysis shows that continuous dependence on the initial condition fails at $\mathbf{x}_0 \in \mathcal{M}$. The following example describes this behavior, and also an analysis of the necessity of removing reset instants when checking the Hausdorff distance (31).

Example IV.1: Consider the reset control system $(A, C, 1)$ with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (41)$$

The reset set is $\mathcal{M} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\} \setminus \{(0, 0)\}$. For any $\mathbf{x}_0 \neq \mathbf{0}$ there is only one reset instant, and thus $\mathbb{T} = (t_1, \infty)$. By simplicity, consider $\mathbf{x}_0 = (1, 0)$ that results in $t_1 = \frac{\pi}{2}$, and the perturbed initial condition $\mathbf{x}_0^* = (1 + \delta \cos(\phi), \delta \sin(\phi))$, for some $\delta > 0$ and $\phi \in [0, 2\pi)$, satisfying $\|\mathbf{x}_0 - \mathbf{x}_0^*\| < \delta$, that results in a resetting instant $t_1^* \in [\frac{\pi}{2} - \text{atan } \delta, \frac{\pi}{2} + \text{atan } \delta]$. Two cases are separately analyzed: $T < t_1$ and $T > t_1$. For $T < t_1 = \frac{\pi}{2}$ it is always possible to find some $\delta_1 > 0$ such as $T < \frac{\pi}{2} - \text{atan } \delta_1$ meaning that both the solution and the perturbed solution always reset after the instant T ; thus, both solutions correspond to the base system solutions and directly $d_H(\mathbf{x}^*([0, T], \mathbf{x}_0^*), \mathbf{x}([0, T], \mathbf{x}_0)) \leq \delta_1$. For the case $T > t_1$, it is also possible to find some δ_2 such as $\frac{\pi}{2} + \text{atan } \delta_2 < T$, that is both the solution and the perturbed solution always reset before the instant T , and again $d_H(\mathbf{x}^*([0, T], \mathbf{x}_0^*), \mathbf{x}([0, T], \mathbf{x}_0)) \leq \delta_2$. On the other hand, note that for $T = t_1 = \frac{\pi}{2}$ it is not possible to choose any $\delta > 0$ such as $T = \frac{\pi}{2} < \frac{\pi}{2} - \text{atan } \delta$ or $T = \frac{\pi}{2} > \frac{\pi}{2} + \text{atan } \delta$, meaning that for $T = t_1 = \frac{\pi}{2}$, there always exist perturbed solutions that have performed resets at instants $t_1^* < T$ and thus $\mathbf{0} \in \mathbf{x}^*([0, \frac{\pi}{2}], \mathbf{x}_0^*)$, while on the other hand $\mathbf{0} \notin \mathbf{x}([0, \frac{\pi}{2}], \mathbf{x}_0)$ and thus there exists perturbed solutions for which $d_H(\mathbf{x}^*([0, \frac{\pi}{2}], \mathbf{x}_0^*), \mathbf{x}([0, \frac{\pi}{2}], \mathbf{x}_0)) > 1$. This is the reason why resetting instants corresponding to \mathbf{x}_0 needs to be removed when checking Hausdorff distances in (31) according to Def IV.1. On the other hand, for $\mathbf{x}_0 = (0, 1) \in \mathcal{M}$ (see Fig. 11.b) the perturbed initial condition $\mathbf{x}_0^* = (\delta, 1)$, and $T = t_1^*$ (the first crossing instant corresponding to \mathbf{x}_0^*), it is true that $\|\mathbf{x}_0 - \mathbf{x}_0^*\| = \delta$, and $d_H(\mathbf{x}([0, t_1^*], \mathbf{x}_0), \mathbf{x}^*([0, t_1^*], \mathbf{x}_0^*)) \geq d_H(\{\mathbf{0}\}, \mathbf{x}^*([0, t_1^*], \mathbf{x}_0^*)) = \sqrt{1 + \delta^2} > 1$ for any $\delta > 0$. As a result, the solution does not depend continuously on the initial condition at any $\mathbf{x}_0 \in \mathcal{M}$.

Another source of problems arises when there are reset system trajectories that are *tangential* to the reset set. More specifically, for a reset control system (A, C, n_ρ) there exist a *tangential crossing* of a solution \mathbf{x} with initial condition \mathbf{x}_0 , at the instant $t > 0$, if $\mathbf{x}(t) \in \mathcal{M}$ and $C\dot{\mathbf{x}}(t) = 0$. In addition, a crossing that it is not tangential will be referred to as a *transversal crossing*.

Example IV.2: The reset control system $(A, C, 1)$, with

$$A = \begin{pmatrix} 0 & -3 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \quad (42)$$

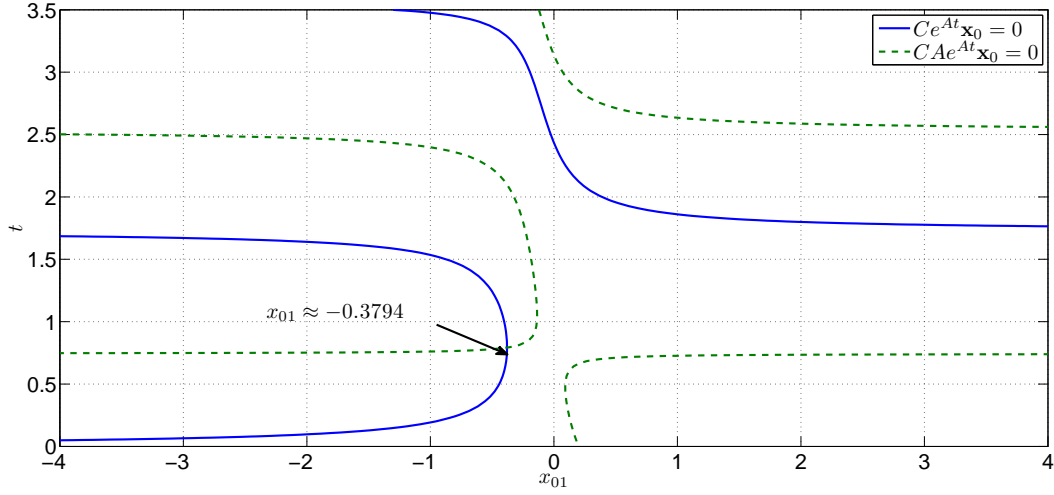


Fig. 12. Zero-level curves and tangential crossing at $x_{01} \approx -0.3794$.

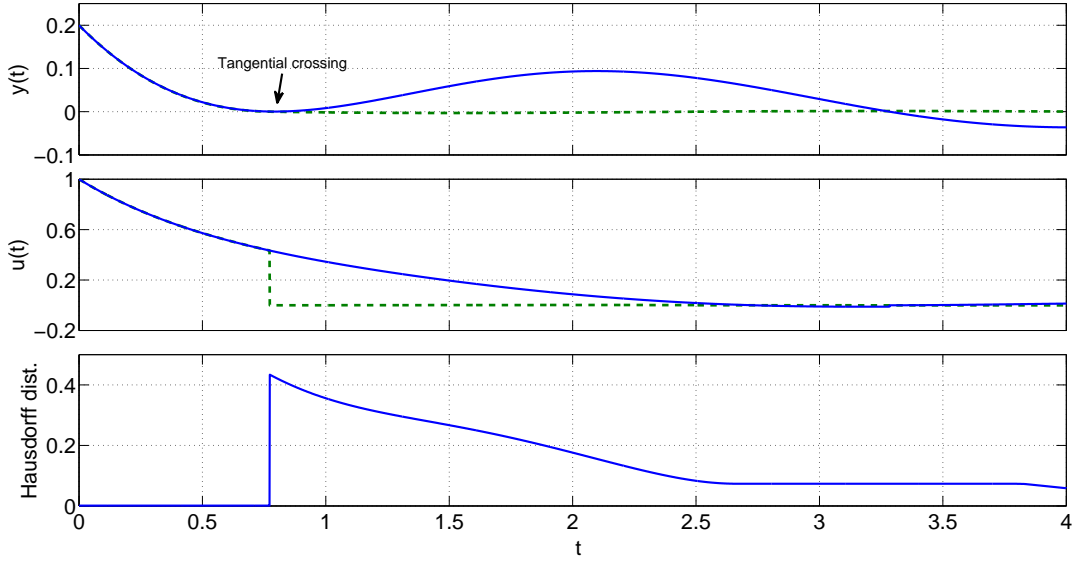


Fig. 13. Time simulation and Hausdorff distance: (dotted) tangential crossing at $t_1 \approx 0.7926$, (solid) no crossing at $t_1 \approx 0.7926$.

has a tangential crossing with \mathcal{M} at the first crossing for $\mathbf{x}_0 = (x_{01}, 0.2, 1)$, where $x_{01} \approx -0.3794$, and for $t_1 \approx 0.7926$. These values can be obtained by numerically solving $C e^{At_1} \mathbf{x}_0 = 0$ and $C A e^{At_1} \mathbf{x}_0 = 0$ for t_1 . Note that $\mathbf{x}_1 = e^{At_1} \mathbf{x}_0 \in \mathcal{M}$, whereas $\mathbf{x}_1 \approx (0, 0, 0.4258)$. Fig. 12 shows the zero-level curves of both $C e^{At_1} \mathbf{x}_0$ and $C A e^{At_1} \mathbf{x}_0$ in the plane (x_{01}, t_1) , clearly showing that there is only one solution for $\mathbf{x}_0 = (x_{01}, 0.2, 1)$ corresponding to $x_{01} \approx -0.3794$. In Fig. 13 two solutions of the reset system have been plotted for the initial condition \mathbf{x}_0 and a perturbed initial condition $\mathbf{x}_0^* = (x_{01} + \delta, 0.2, 1)$ for some small value $\delta > 0$. Note that \mathbf{x}_0^* does not produce a crossing and thus no reset action is performed at the instant t_1 . The two upper plots show the system output $y(t) = C\mathbf{x}(t)$ and the reset state $x_3(t)$, while the bottom plot shows the Hausdorff distance $d_H(\mathbf{x}([0, t], \mathbf{x}_0), \mathbf{x}^*([0, t], \mathbf{x}_0^*))$ versus t . It turns out that for $t > t_1$, $d_H(\mathbf{x}([0, t], \mathbf{x}_0), \mathbf{x}^*([0, t], \mathbf{x}_0^*))$ can not be made arbitrarily small by making δ small enough, and thus the solution does not depend continuously on the initial condition at $\mathbf{x}_0 = (x_{01}, 0.2, 1)$.

B. Crossing instants versus reset instants

Besides reset instants, crossings of $\mathcal{H}_C (= \bar{\mathcal{M}})$ play an important role in the continuous dependence analysis of reset control systems.

Definition IV.2 (well-posed crossing instants): A reset control system (A, C, n_ρ) has well-posed crossing instants if for any $\mathbf{x}_0 \in \mathbb{R}^n$ there exists a sequence $\bar{\mathbb{T}} : \{1, 2, \dots\} \rightarrow \mathbb{R}^+ \cup \{\infty\}$, denoted by $\bar{\mathbb{T}} = (\bar{t}_1, \bar{t}_2, \dots)$, and given by (being $\bar{t}_0 = 0$):

- $\bar{t}_1 = \begin{cases} 0 & , \mathbf{x}_0 \in \bar{\mathcal{M}} \\ \min\{\Delta \in \mathbb{R}^+ : e^{A\Delta}\mathbf{x}_0 \in \bar{\mathcal{M}}\} & , \mathbf{x}_0 \in \mathbb{R}^n \setminus \bar{\mathcal{M}} \end{cases}$
- $i = 1$
while $\bar{t}_i \neq \infty$ **do**
 if $\mathbf{x}_{i-1} \in \mathcal{M}_{RU}$ **then**
 $\bar{t}_{i+1} = \infty$
 else
 $\mathbf{x}_i = A_R e^{A(\bar{t}_i - \bar{t}_{i-1})} \mathbf{x}_{i-1}$
 $\bar{t}_{i+1} = \bar{t}_i + \min\{\Delta > 0 : e^{A\Delta}\mathbf{x}_i \in \bar{\mathcal{M}}\}$
 $i \leftarrow i + 1$
 $N = i$
 end if
end while

Note that $\bar{\mathbb{T}} = (\infty)$ corresponds to a solution without crossings of $\bar{\mathcal{M}}$, $\bar{\mathbb{T}} = (\bar{t}_1, \infty)$ to a solution with a unique crossing at the instant \bar{t}_1, \dots . Note that since $\mathcal{M} \subset \bar{\mathcal{M}}$ then \mathbb{T} is a subsequence of $\bar{\mathbb{T}}$; and that $\bar{\mathbb{T}} = (0, \infty)$ for $\mathbf{x}_0 \in \mathcal{M}_{RU} \subset \bar{\mathcal{M}}$.

Example IV.3: The reset control system of Example IV.1 (Fig. 8) has the crossing instants sequence $\bar{\mathbb{T}} = \mathbb{T} = (t_1, \infty)$, with $\bar{t}_1 = t_1$, for $\mathbf{x}_0 \neq 0$; and $\bar{\mathbb{T}} = (0, \infty)$, $\mathbb{T} = (\infty)$, for $\mathbf{x}_0 = 0$ (note that $\mathcal{M}_{RU} = \{0\}$).

Example IV.4: For the reset control system of Example IV.2, consider an initial condition $(1, x_{02}, 0)$, where $x_{02} \in \mathbb{R}$. In Fig. 14, the solutions (crossing instants) of $Ce^{A\bar{t}}(1, x_{02}, 0) = 0$ for \bar{t} has been plotted vs. x_{02} . To decide if a crossing instant is also a reset instant it has to be checked if the state corresponding to that instant belong to \mathcal{M}_R or not. For $x_{02} = 0$, it result that $\bar{t}_1 = 0$, $\bar{t}_2 = t_1$, $\bar{t}_3 = t_2, \dots$; and for $x_{20} < 0$, $\bar{t}_1^* = t_1^*$, $\bar{t}_2^* = t_2^*, \dots$ (t_1, t_2, t_1^* , and t_2^* are shown in Fig.11).

Proposition IV.1: A reset control system (A, C, n_ρ) with well-posed reset instants has well-posed crossing instants, that is there exists $\bar{\mathbb{T}} = (\bar{t}_1, \bar{t}_2, \dots)$ for any $\mathbf{x}_0 \in \mathbb{R}^n$, and in addition $0 \leq \bar{t}_1 < \bar{t}_2 < \dots$, and $\bar{t}_k \rightarrow \infty$ as $k \rightarrow \infty$ if the sequence is infinite.

Proof: It follows similar arguments to proof of Prop. III.2-III.3 and is omitted by brevity. \square

Besides functions $\tau_i, i = 1, 2, \dots$ mapping \mathbf{x}_0 to the i^{th} reset instant, functions $\bar{\tau}_i : \mathbb{R}^n \rightarrow \mathbb{R}^+$ may be defined such as $\bar{t}_i = \bar{\tau}_i(\mathbf{x}_0), i = 1, 2, \dots$. Even in simple cases, like the third order reset control system of Example IV.2, it is true that they have jump discontinuities. In that example (see Fig. 11), for an x_{20} arbitrary close to zero and $x_{20} < 0$, note that $\mathbb{T} = (t_1^*, t_2^*, t_3^*, \dots)$, where $t_1^* > 0$ is arbitrarily close to 0, t_2^* is arbitrarily close to t_1 , t_3^* is arbitrarily close to t_2, \dots ; and for $x_{20} > 0$, $\mathbb{T} = (t_1^{**}, t_2^{**}, \dots)$ (see Fig. 11), with t_1^{**} arbitrarily close to t_1 , t_2^{**} arbitrarily close to t_2, \dots). Thus any $\tau_i, i = 1, 2, \dots$ is discontinuous at $(1, 0, 0)$. Obviously, since the reset instants sequence is a subsequence of the crossing instants sequence, then this is also the case for $\bar{\tau}_i$. In spite of the lack of regularity of the reset instants pattern, some simple properties of the crossing instants sequences have been discovered. These properties will be key to derive a sufficient condition for continuous dependence on the initial condition.

Proposition IV.2: Let (A, C, n_ρ) be a reset control system with well-posed reset instants and $\mathbf{x}_0 \in \mathbb{R}^n \setminus \bar{\mathcal{M}}$. If $\bar{t}_1 < \infty$ and $CA\mathbf{x}(\bar{t}_1, \mathbf{x}_0) \neq 0$ then $\bar{\tau}_1$ is continuous at \mathbf{x}_0 .

Proof: It is a direct consequence of the implicit function theorem. Consider the continuously differentiable function $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f(\mathbf{x}, t) = Ce^{At}\mathbf{x}$, where $f(\mathbf{x}_0, t_1) = 0$ and in addition

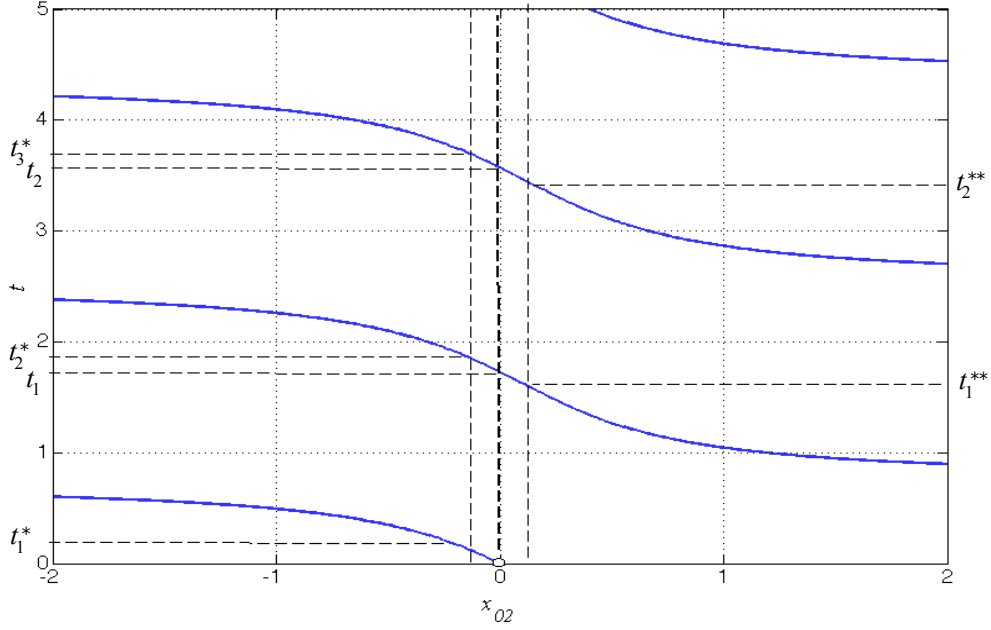


Fig. 14. Zero-level curves and reset instants: $(x_{02} = 0) \mathbb{T} = (t_1, t_2, \dots)$, $(x_{02} < 0) \mathbb{T}^* = (t_1^*, t_2^*, \dots)$, $(x_{02} > 0) \mathbb{T}^{**} = (t_1^{**}, t_2^{**}, \dots)$.

$(\frac{\partial f}{\partial t})(\mathbf{x}_0, t_1) = CAe^{At_1}\mathbf{x}_0 = CA\mathbf{x}(t_1, \mathbf{x}_0) \neq 0$. Then, there exists open sets $U \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$, with $\mathbf{x}_0 \in U$ and $t_1 \in I$, and a (unique) continuously differentiable function $\bar{\tau}_1 : U \rightarrow I$ such as $0 = f(\mathbf{x}_0, \bar{\tau}_1(\mathbf{x}_0)) = Ce^{A\bar{\tau}_1(\mathbf{x}_0)}\mathbf{x}_0$. \square

Proposition IV.3: Let (A, C, n_ρ) be a reset control system with well-posed reset instants and $\mathbf{x}_0 \in \mathcal{M}_R \setminus \mathcal{M}_{RU}$ (and thus $\bar{t}_1 = 0$). If $\bar{t}_2 < \infty$ and $CA\mathbf{x}(\bar{t}_k, \mathbf{x}_0) \neq 0$, $k = 1, 2$ then there exist $d_0, d_I > 0$ such that $\mathbf{x}^*([0, \bar{t}_2 + d_I], \mathbf{x}_0^*)$, with $\|\mathbf{x}_0 - \mathbf{x}_0^*\| < d_0$, intersects the set $\bar{\mathcal{M}}$ either:

- at the instant $\bar{t}_1^* \in (\bar{t}_2 - d_I, \bar{t}_2 + d_I)$, and for any $\alpha > 0$ there exists some $\delta > 0$ such as if $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta$ then $|\bar{t}_1^* - \bar{t}_2| < \alpha$, or
- at the instants $\bar{t}_1^* \in [0, d_I)$ and $\bar{t}_2^* \in (\bar{t}_2 - d_I, \bar{t}_2 + d_I)$, and for any $\alpha > 0$ there exists some $\delta > 0$ such as if $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta$ then $\bar{t}_1^* < \alpha$ and $|\bar{t}_2^* - \bar{t}_2| < \alpha$.

Proof: Consider again the continuously differentiable function f such as $f(\mathbf{x}, t) := Ce^{At}\mathbf{x}$, and thus $f(\mathbf{x}_0, 0) = f(\mathbf{x}_0, \bar{t}_2) = 0$. Assume that $f(\mathbf{x}_0, t) > 0$ for all $t \in (0, \bar{t}_2)$, otherwise a similar case is obtained. In addition, it is possible to find constants $\epsilon > 0$, and $\alpha_1(\epsilon), \alpha_2(\epsilon) > 0$ such that:

- $f(\mathbf{x}_0, \alpha_1) = f(\mathbf{x}_0, \bar{t}_2 - \alpha_2) = \epsilon$ with $\alpha_1 + \alpha_2 < \bar{t}_2$.
- $f(\mathbf{x}_0, t)$ is monotone increasing with respect to $t \in (0, \alpha_1)$ and monotone decreasing with respect to $t \in (\bar{t}_2 - \alpha_2, \bar{t}_2)$.
- $f(\mathbf{x}_0, t) > \epsilon$ for all $t \in (\alpha_1, \bar{t}_2 - \alpha_2)$.

Also, since f is continuous at $(\mathbf{x}_0, t) \in \mathbb{R}^n \times \mathbb{R}^+$ then there exists $\delta(\epsilon, \bar{t}_2)$ such that for all $\mathbf{x}_0^* \in \mathbb{B}(\mathbf{x}_0, \delta)$ and $t \in (\alpha_1, \bar{t}_2 - \alpha_2)$: $0 < f(\mathbf{x}_0, t) - \epsilon < f(\mathbf{x}_0^*, t) < f(\mathbf{x}_0, t) + \epsilon$. Now, since $\frac{\partial f}{\partial t}(\mathbf{x}(\bar{t}_2, \mathbf{x}_0), \bar{t}_2) = CA\mathbf{x}(\bar{t}_2, \mathbf{x}_0) \neq 0$ then, in virtue of the implicit function theorem, there exist constants $\delta_1 > 0$ and $\alpha > 0$ such that $\mathbf{x}^*([\bar{t}_2 - \alpha, \bar{t}_2 + \alpha], \mathbf{x}_0^*)$ intersects $\bar{\mathcal{M}}$ for any $\mathbf{x}_0^* \in \mathbb{B}(\mathbf{x}_0, \delta_1)$, and as a consequence. $\bar{t}_1^* < \bar{t}_2 + \alpha$

Property (i) can be rewritten by defining a constant $\alpha_3(\epsilon)$ such that $f(\mathbf{x}_0, \bar{t}_2 + \alpha_3) = -\epsilon$. From (i), (ii) and (iii) it is concluded that $\bar{t}_1^* \in (0, \alpha_1) \cup (\bar{t}_2 - \alpha_2, \bar{t}_2 + \alpha_3)$. Also due to $\lim_{\epsilon \rightarrow 0} A_R e^{At_1^*} \mathbf{x}_0 = \mathbf{x}_0$, it is possible to find ϵ_0 such that for all $\epsilon < \epsilon_0$, $\|A_R e^{At_1^*} \mathbf{x}_0 - \mathbf{x}_0\| < \delta$ and then $\bar{t}_2^* \in (0, \alpha_1) \cup (\bar{t}_2 - \alpha_2, \bar{t}_2 + \alpha_3)$.

Now, it is proven that if $\bar{t}_1^* \in (0, \alpha_1)$ then $\bar{t}_2^* \notin (0, \alpha_1)$. By contradiction, assume that $\bar{t}_1^* \in (0, \alpha_1)$ and $\bar{t}_2^* \in (0, \alpha_1)$, then in virtue of the Rolle theorem there exists $s \in (0, \bar{t}_2)$ such that $\frac{\partial f}{\partial t}(\mathbf{x}(s, \mathbf{x}_0), s) = 0$. It is obvious that $\mathbf{x}_0^* \rightarrow \mathbf{x}_0$ as $\epsilon \rightarrow 0$ and then from (ii) $\alpha_1 \rightarrow 0$ as $\epsilon \rightarrow 0$, which means that $s \rightarrow 0$ as $\epsilon \rightarrow 0$ and finally $CA\mathbf{x}_0 = 0$, which is in contradiction with one of the assumptions in the Proposition.

Finally, (ii) implies that $\alpha_k \rightarrow 0$ as $\epsilon \rightarrow 0$, $k = 1, 2$. For an arbitrary $\alpha > 0$ there exists $\epsilon_1 < \epsilon_0$ such that $\alpha_k(\epsilon) < \alpha$ for $\epsilon < \epsilon_1$ and such that (iii) holds. Then the result follows immediately since $\bar{t}_1^* \in (0, \alpha) \cup (\bar{t}_2 - \alpha, \bar{t}_2 + \alpha)$, and if $\bar{t}_1^* \in (0, \alpha)$ then $\bar{t}_2^* \in (\bar{t}_2 - \alpha, \bar{t}_2 + \alpha)$. In particular there exist constants $d_I = \alpha$ and $d_0 = \delta(\epsilon)$ satisfying the Proposition. \square

C. A sufficient condition for continuous dependence on the initial condition

Since continuous dependence is not possible to obtain for any arbitrary initial condition (for example, initial conditions in the reset set \mathcal{M}), the problem is to characterize the set D of initial conditions that have the property. As a result, $D \subset \mathbb{R}^n \setminus \mathcal{M}$. In addition, initial conditions that produce tangential crossings (see Example IV.2) must be excluded.

In the following, several basic results to be used in the next Proposition will be derived. Firstly, consider a solution $\mathbf{x}(t)$ on $[s_0, s_1]$, such that $t_k \notin [s_0, s_1]$, $k = 1, 2, \dots$, and with $\|\mathbf{x}(s_0)\| < M$ for some constant $M > 0$, then:

- $\|\mathbf{x}(t) - \mathbf{x}(s_0)\| \leq \|e^{A(t-s_0)}\mathbf{x}(s_0) - \mathbf{x}(s_0)\| \leq (e^{L(s_1-s_0)} - 1)M$, where L is the spectral norm of A . Thus, $\beta_1(\cdot, M) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined as

$$\beta_1(\epsilon, M) := \frac{1}{L} \ln(1 + \frac{\epsilon}{M}) \quad (43)$$

and $s_1 - s_0 < \alpha = \beta_1(\epsilon, M) \Rightarrow \|\mathbf{x}(t) - \mathbf{x}(s_0)\| < \epsilon$ for $t \in [s_0, s_1]$.

- (CDBS property) $\|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq e^{L(s_1-s_0)}\|\mathbf{x}^*(s_0) - \mathbf{x}(s_0)\|$. $\beta_2(\cdot, s_0, s_1) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined as

$$\beta_2(\epsilon, s_0, s_1) := \frac{\epsilon}{e^{L(s_1-s_0)}} \quad (44)$$

Thus, if $\|\mathbf{x}^*(s_0) - \mathbf{x}(s_0)\| < \delta = \beta_2(\epsilon, s_0, s_1)$ then $\|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq \epsilon$ for $t \in [s_0, s_1]$

In addition, if conditions of Prop. IV.2 are satisfied then it is possible to make $|t_1^* - t_1| < \alpha$ by doing $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta$, and $\delta > 0$ small enough. Thus, a map $\beta_3 : [0, d_I] \rightarrow \mathbb{R}^+$ is defined such as $\delta = \beta_3(\alpha)$ is the largest δ with that property. And finally, if Prop. IV. 3 applies then by doing $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta$, and $\delta > 0$ small enough, in the cases there are intersections of $\bar{\mathcal{M}}$ either $|\bar{t}_1^* - \bar{t}_2| < \alpha$ or $\bar{t}_1^* < \alpha$ and $|\bar{t}_2^* - \bar{t}_2| < \alpha$; now, a map $\beta_4 : [0, d_I] \rightarrow \mathbb{R}^+$ is defined such as $\delta = \beta_4(\alpha)$ is the largest δ with that property.

Proposition IV.4: For a reset control system (A, C, n_ρ) with well-posed reset instants, the solution depends continuously on the initial condition at $\mathbf{x}_0 \in \mathbb{R}^n \setminus \mathcal{M}$ if $CA\mathbf{x}(t_k, \mathbf{x}_0) \neq 0$, $k = 1, 2, \dots$.

Proof: Since (A, C, n_ρ) has well-posed reset instants and Prop. IV.1 applies, then there exists reset and crossing instants sequences $\mathbb{T} = (t_i)$ and $\bar{\mathbb{T}} = (\bar{t}_i)$, $i = 1, 2, \dots$, with $0 \leq t_1 < t_2 < \dots$ and $0 \leq \bar{t}_1 < \bar{t}_2 < \dots$, and also there exists a unique solution $\mathbf{x}(t)$ for any $t \geq 0$. Two cases, $\mathbf{x}_0 \notin \mathcal{M}_R$ (that is $\mathbf{x}_0 \notin \bar{\mathcal{M}}$ since $\mathbf{x}_0 \in \mathbb{R}^n \setminus \mathcal{M}$ by assumption) and $\mathbf{x}_0 \in \mathcal{M}_R$, will be separately treated. In any case, there exist a constant $M_0 > 0$ such as $\|\mathbf{x}_0\| \leq M_0$.

Case A ($\mathbf{x}_0 \notin \mathcal{M}_R$). It will be assumed that $\bar{t}_1 < \infty$, otherwise the result directly follows from the CDBS property. If $A_R\mathbf{x}(\bar{t}_1) \notin \mathcal{M}_{RU}$, then $\mathbf{x}(t) \notin \bar{\mathcal{M}}$ for any $t \in (\bar{t}_1, \bar{t}_2)$ and in this case the proof will be based on checking if $d_H(\mathbf{x}^*([0, T]), \mathbf{x}([0, T]))$ is arbitrarily small when \mathbf{x}_0^* and \mathbf{x}_0 are arbitrarily close, for $0 \leq T < \bar{t}_{12}$ and some intermediate instant $\bar{t}_{12} \in (\bar{t}_1, \bar{t}_2)$. If $\bar{t}_2 = \infty$ then the proof will be finished; otherwise, note that since $\mathbf{x}(\bar{t}_{12}) \notin \mathcal{M}_R$ (in fact $\mathbf{x}(\bar{t}_{12}) \notin \bar{\mathcal{M}}$) then $\mathbf{x}(\bar{t}_{12})$ can be redefined as a new initial condition belonging to this Case A, and thus a similar argument may be used to analyze $d_H(\mathbf{x}^*([0, T]), \mathbf{x}([0, T]))$ for $\bar{t}_{12} \leq T < \bar{t}_{23}$, and some $\bar{t}_{23} \in (\bar{t}_2, \bar{t}_3)$, etc. The cases $\mathbf{x}(\bar{t}_1) \in \mathcal{M}$ and $\mathbf{x}(\bar{t}_1) \notin \mathcal{M}$ (that is $\mathbf{x}(\bar{t}_1) \in \mathcal{M}_R$), will be separately treated as Cases A.1 and A.2. Finally, if $A_R\mathbf{x}(\bar{t}_1) \in \mathcal{M}_{RU}$ then $\mathbf{x}(t) \in \mathcal{M}_{RU}$ for any $t \in (\bar{t}_1, \bar{t}_2) = (\bar{t}_1, \infty)$, since \mathcal{M}_{RU} is A -invariant by well-posedness of the reset instants; this will be the Case A.3.

Case A.1 ($\mathbf{x}_0 \notin \mathcal{M}_R$, $\mathbf{x}(\bar{t}_1) \in \mathcal{M}$, $A_R\mathbf{x}(\bar{t}_1) \notin \mathcal{M}_{RU}$): Here $0 < \bar{t}_1 = t_1 < \bar{t}_2$, and by Prop. IV.2 it is true that the perturbed trajectory $\mathbf{x}^*([0, t_1 + d_I], \mathbf{x}_0^*)$ intersects the set \mathcal{M} at some instant $\bar{t}_1^* = t_1^* \in (t_1 - \alpha, t_1 + \alpha)$,

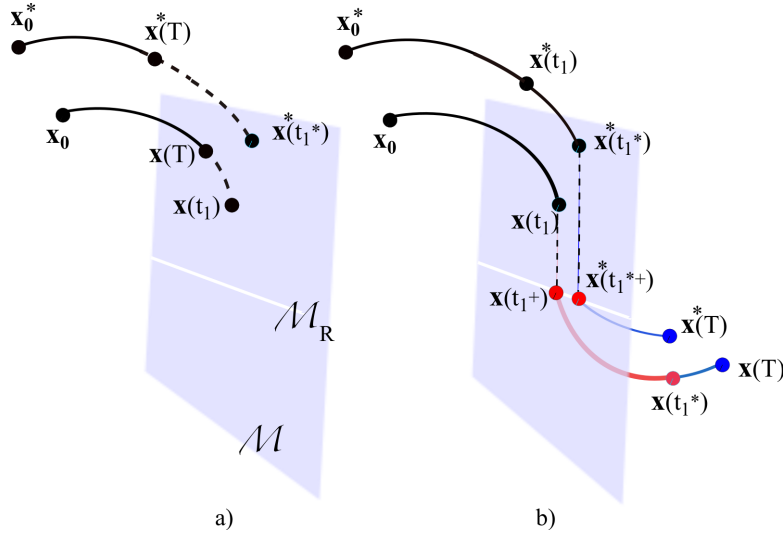


Fig. 15. Case A.1 ($t_1 < t_1^*$): a) A.1.1: $T < t_1(< t_1^*)$, b) A.1.2: $t_1(< t_1^*) < T < \bar{t}_2$

where $\alpha \in [0, d_I)$ for $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta$ and $\delta = \beta_3(\alpha)$. Now, $d_H(\mathbf{x}^*([0, T], \mathbf{x}_0^*), \mathbf{x}([0, T], \mathbf{x}_0))$ is checked for $T \in [0, \bar{t}_2)$ and $T \neq t_1$.

– *Case A.1.1:* For $0 \leq T < t_1(= \bar{t}_1)$ (Fig 15.a), choose $\alpha > 0$ such as $T < t_1 - \alpha$, meaning that the solution and any perturbed solution intersects the reset set after the instant T , and all the trajectories are given by the base system. Now, for $\delta = \min\{\beta_2(\epsilon, 0, T), \beta_3(\alpha)\}$ it is true that $\|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq \epsilon$ for $t \in [0, T]$, and thus it directly follows that $\|\mathbf{x}_0 - \mathbf{x}_0^*\| < \delta \Rightarrow d_H(\mathbf{x}^*([0, T], \mathbf{x}_0^*), \mathbf{x}([0, T], \mathbf{x}_0)) < \epsilon$.

– *Case A.1.2:* For $t_1 < T < \bar{t}_{12}$ (Fig. 15.b), choose $\alpha_1 > 0$ such as $t_1 + \alpha_1 < T$, and thus any perturbed trajectory also intersects the reset set at an instant $t_1^* < T$. Now, assume that $t_1 < t_1^*$ (otherwise a similar reasoning may be applied), thus $t_1 < t_1^* < T$. Firstly, distances between trajectories and perturbed trajectories will be bounded in some intervals, then these bounds will be used to obtain bounds of the Hausdorff distance.

- $0 \leq t \leq t_1$: Directly, by the CDBS property $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta_1 = \min\{\beta_2(\epsilon_1, 0, t_1), \beta_3(\alpha_1)\} \Rightarrow \|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq \epsilon_1$. In addition, $\|\mathbf{x}(t_1)\| = \|e^{At_1}\mathbf{x}_0\| \leq e^{Lt_1}M_0 =: M_1$, and $\|\mathbf{x}^*(t_1)\| \leq \|\mathbf{x}^*(t_1) - \mathbf{x}(t_1)\| + \|\mathbf{x}(t_1)\| \leq \epsilon_1 + M_1 =: M_1^*$.
- $t_1 < t \leq t_1^*$: By making $\delta_2 = \min\{\beta_3(\beta_1(\frac{\epsilon_2}{2}, M_1^*)), \delta_1\}$ it is true that $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta_2 \Rightarrow$

$$\begin{aligned} \|\mathbf{x}^*(t) - \mathbf{x}(t_1)\| &\leq \|\mathbf{x}^*(t) - \mathbf{x}^*(t_1)\| + \\ &+ \|\mathbf{x}^*(t_1) - \mathbf{x}(t_1)\| \leq \frac{\epsilon_2}{2} + \epsilon_1 \leq \epsilon_2 \end{aligned} \quad (45)$$

by doing $\epsilon_1 \leq \epsilon_2/2$. In addition, $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta_3 = \min\{\beta_3(\beta_1(\frac{\epsilon_3}{2}, M_1)), \delta_2\} \Rightarrow$

$$\begin{aligned} \|\mathbf{x}(t) - \mathbf{x}^*(t_1^{*+})\| &\leq \|\mathbf{x}(t) - \mathbf{x}(t_1^+)\| + \|\mathbf{x}(t_1^+) - \mathbf{x}^*(t_1^{*+})\| \\ &< \frac{\epsilon_3}{2} + \|A_R\mathbf{x}(t_1) - A_R\mathbf{x}^*(t_1^*)\| \\ &< \frac{\epsilon_3}{2} + \|\mathbf{x}(t_1) - \mathbf{x}^*(t_1^*)\| < \frac{\epsilon_3}{2} + \epsilon_2 \leq \epsilon_3 \end{aligned} \quad (46)$$

where it has been used the fact that $\|A_R\| = 1$, and chosen $\epsilon_2 \leq \epsilon_3/2$.

- $t_1^* < t \leq T$: By doing $\epsilon_3 = \beta_2(\epsilon_4, t_1^{*+}, T)$, and since $\mathbf{x}(t_1^{*+}) = \mathbf{x}(t_1^*)$ then by using the CDBS property, $\|\mathbf{x}^*(t_1^{*+}) - \mathbf{x}(t_1^{*+})\| < \epsilon_3 \Rightarrow \|\mathbf{x}^*(t) - \mathbf{x}(t)\| < \epsilon_4$.

Now, the Hausdorff distance between the sets $\mathbf{x}^*([0, T])$ and $\mathbf{x}([0, T])$ will be bounded using the above bounds. Firstly, since

$$d_E(\mathbf{x}^*(t), \mathbf{x}([0, T])) \leq \begin{cases} d_E(\mathbf{x}^*(t), \mathbf{x}([0, t_1])) \leq \epsilon_1, & 0 \leq t \leq t_1 \\ d_E(\mathbf{x}^*(t), \mathbf{x}([t_1, t_1^*])) \leq \epsilon_3, & t_1 \leq t \leq t_1^* \\ d_E(\mathbf{x}^*(t), \mathbf{x}([t_1^*, T])) \leq \epsilon_4, & t_1^* \leq t \leq T \end{cases} \quad (47)$$

then $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta_3 \Rightarrow$

$$h(\mathbf{x}([0, T]), \mathbf{x}^*([0, T])) \leq \epsilon_4 \quad (48)$$

Finally, since

$$d_E(\mathbf{x}(t), \mathbf{x}^*([0, T])) \leq \begin{cases} d_E(\mathbf{x}(t), \mathbf{x}^*([0, t_1])) \leq \epsilon_1, & 0 \leq t \leq t_1 \\ d_E(\mathbf{x}(t), \mathbf{x}^*([t_1, t_1^*])) \leq \epsilon_2, & t_1 \leq t \leq t_1^* \\ d_E(\mathbf{x}(t), \mathbf{x}^*([t_1^*, T])) \leq \epsilon_4, & t_1^* \leq t \leq T \end{cases} \quad (49)$$

then $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta_3 \Rightarrow$

$$h(\mathbf{x}^*([0, T]), \mathbf{x}([0, T])) \leq \epsilon_4 \quad (50)$$

and using (12) it is concluded that for a given $\epsilon := \epsilon_4$ there exist $\delta := \delta_3$ such as

$$\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta \Rightarrow d_H(\mathbf{x}^*([0, T]), \mathbf{x}([0, T])) \leq \epsilon \quad (51)$$

Case A.2 ($\mathbf{x}_0 \notin \mathcal{M}_R$, $\mathbf{x}(\bar{t}_1) \in \mathcal{M}_R$): Here $0 < \bar{t}_1 < \bar{t}_2 \leq t_1$, and by Prop. IV.2 the perturbed trajectory $\mathbf{x}^*([0, \bar{t}_1 + d_I], \mathbf{x}_0^*)$ intersects the set $\bar{\mathcal{M}}$ at instant $\bar{t}_1^* \in (\bar{t}_1 - \alpha, \bar{t}_1 + \alpha)$, where $\alpha > 0$ is arbitrarily small for $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta$ and $\delta > 0$ arbitrarily small.

– *Case A.2.1*: For $0 < T < \bar{t}_1$, choose $\alpha > 0$ such as $T < \bar{t}_1 - \alpha < \bar{t}_1^*$, and thus $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta = \min\{\beta_2(\epsilon, 0, \bar{t}_1), \beta_3(\alpha)\} \Rightarrow d_H(\mathbf{x}^*([0, T], \mathbf{x}_0^*), \mathbf{x}([0, T], \mathbf{x}_0)) < \epsilon$ directly follows from the CDBS property.

– *Case A.2.2*: For $\bar{t}_1 \leq T < \bar{t}_{12}$ assume that $\bar{t}_1^* < \bar{t}_1$, otherwise a similar reasoning may be applied; thus,

- $0 \leq t \leq \bar{t}_1^*$: Again choose $\alpha_1 > 0$ such as $T < \bar{t}_1 - \alpha_1 < \bar{t}_1^*$ and thus $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta_1 = \min\{\beta_2(\epsilon_1, 0, \bar{t}_1), \beta_3(\alpha_1)\} \Rightarrow \|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq \epsilon_1$. In addition, $\|\mathbf{x}(\bar{t}_1^*)\| = \|e^{A\bar{t}_1^*} \mathbf{x}_0\| \leq e^{L\bar{t}_1^*} M_0 \leq e^{L\bar{t}_1} M_0 =: M_1$, and $\|\mathbf{x}^*(\bar{t}_1^*)\| \leq \|\mathbf{x}^*(\bar{t}_1^*) - \mathbf{x}(\bar{t}_1^*)\| + \|\mathbf{x}(\bar{t}_1^*)\| \leq \epsilon_1 + M_1 =: M_1^*$.
- $\bar{t}_1^* < t \leq \bar{t}_1$: For some $\epsilon_2 > 0$ and $\epsilon_1 = \epsilon_2/2$, it is true that $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta_2 = \min\{\beta_4(\beta_1(\frac{\epsilon_2}{2}, M_1)), \delta_1\} \Rightarrow$

$$\begin{aligned} \|\mathbf{x}(t) - \mathbf{x}^*(\bar{t}_1^*)\| &\leq \|\mathbf{x}(t) - \mathbf{x}(\bar{t}_1^*)\| + \|\mathbf{x}(\bar{t}_1^*) - \mathbf{x}^*(\bar{t}_1^*)\| \\ &\leq \frac{\epsilon_2}{2} + \epsilon_1 = \epsilon_2 \end{aligned} \quad (52)$$

In addition, since $\mathbf{x}(\bar{t}_1) \in \mathcal{M}_R$, and thus $\mathbf{x}(\bar{t}_1) = A_R \mathbf{x}(\bar{t}_1)$, and $\|A_R\| = 1$, then for some $\epsilon_3 > 0$ and $\epsilon_2 = \epsilon_3/2$, it is true that $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta_3 = \min\{\beta_4(\beta_1(\frac{\epsilon_3}{2}, M_1^*)), \delta_2\} \Rightarrow$

$$\|\mathbf{x}^*(t) - \mathbf{x}(\bar{t}_1)\| \leq \|\mathbf{x}^*(t) - \mathbf{x}^*(\bar{t}_1^{*+})\| \quad (53)$$

$$+ \|\mathbf{x}^*(\bar{t}_1^{*+}) - \mathbf{x}(\bar{t}_1)\| \leq \|\mathbf{x}^*(t) - A_R \mathbf{x}^*(\bar{t}_1^*)\|$$

$$+ \|A_R \mathbf{x}^*(\bar{t}_1^*) - A_R \mathbf{x}(\bar{t}_1)\| \leq \frac{\epsilon_3}{2} + \|\mathbf{x}^*(\bar{t}_1^*) - \mathbf{x}(\bar{t}_1)\| \quad (54)$$

$$\leq \frac{\epsilon_3}{2} + \epsilon_2 = \epsilon_3$$

• $\bar{t}_1 < t \leq T < \bar{t}_2$: By the CDBS property, $\|\mathbf{x}^*(\bar{t}_1) - \mathbf{x}(\bar{t}_1)\| < \epsilon_3 = \beta_2(\epsilon_4, \bar{t}_1, T) \Rightarrow \|\mathbf{x}^*(t) - \mathbf{x}(t)\| < \epsilon_4$. Finally, making a reasoning similar to Case A.1.2, it again follows that for a given $\epsilon =: \epsilon_4$ there exist some $\delta =: \delta_3$ such as

$$\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta \Rightarrow d_H(\mathbf{x}^*([0, T]), \mathbf{x}([0, T])) \leq \epsilon \quad (55)$$

Case A.3 ($\mathbf{x}_0 \notin \mathcal{M}_R$, $\mathbf{x}(\bar{t}_1) \in \mathcal{M}$, $A_R \mathbf{x}(\bar{t}_1) \in \mathcal{M}_{RU}$): Here $0 < \bar{t}_1 = t_1 < \bar{t}_2 = \infty$, and again by Prop. IV.2 it results that the perturbed trajectory $\mathbf{x}^*([0, t_1 + d_I], \mathbf{x}_0^*)$ intersects the set \mathcal{M} at some instant $\bar{t}_1^* = t_1^* \in (t_1 - \alpha, t_1 + \alpha)$, where $\alpha \in [0, d_I]$ for $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta$ and $\delta = \beta_3(\alpha)$. For $T \in [0, t_1]$ the

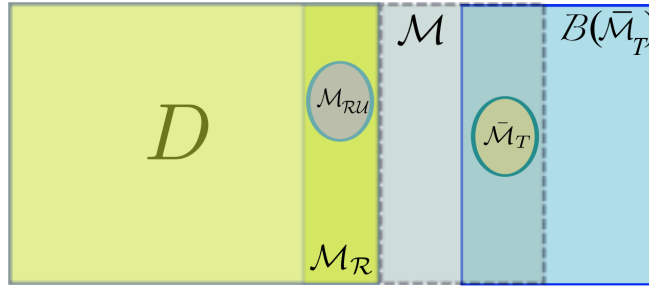


Fig. 16. Set Diagrams for \mathcal{M}_R , \mathcal{M}_{RU} , \mathcal{M} , $\bar{\mathcal{M}}_T$, $\mathcal{B}(\bar{\mathcal{M}}_T)$, and $D = \mathbb{R}^n \setminus (\mathcal{M} \cup \mathcal{B}(\bar{\mathcal{M}}_T))$.

case is identical to Case A.1.1; for $T \in (t_1, \infty)$ the case is similar to Case A.1.2 (assume for example that $t_1 < t_1^* < T$), the difference is that now $t_2 = \infty$ and there may exist (infinitely) many reset instants $t_k^* \in (t_1^*, \infty)$, $k = 2, 3, \dots$. In this case, the result follows from application of the CDBS property and the continuity of the map A_R (note that $\|A_R\| = 1$).

Case B ($\mathbf{x}_0 \in \mathcal{M}_R$). If $\mathbf{x}_0 \in \mathcal{M}_{RU}$ a case similar to case A.3 is obtained. Otherwise, if $\mathbf{x}_0 \in \mathcal{M}_R \setminus \mathcal{M}_{RU}$ then Prop. IV.3 applies and $d_H(\mathbf{x}^*([0, T]), \mathbf{x}([0, T]))$ will be checked to be arbitrarily small, and $\mathbf{x}^*(T)$ and $\mathbf{x}(T)$ arbitrarily close, when \mathbf{x}_0^* and \mathbf{x}_0 are arbitrarily close, for $0 = \bar{t}_1 \leq T < \bar{t}_2$. Once again, the argument is that if that property holds for $T \in [0, \bar{t}_{12})$, where $\bar{t}_{12} \in (0, \bar{t}_2)$, then since $\mathbf{x}(\bar{t}_{12}) \notin \mathcal{M}_R$, in fact $\mathbf{x}(\bar{t}_{12}) \notin \bar{\mathcal{M}}$, then $\mathbf{x}(\bar{t}_{12})$ can be redefined as a new initial condition belonging to Case A, and thus a similar argument may be used to analyze $d_H(\mathbf{x}^*([0, T]), \mathbf{x}([0, T]))$ for $\bar{t}_{12} \leq T < \bar{t}_{23}$, and some $\bar{t}_{23} \in (\bar{t}_2, \bar{t}_3)$, etc. The proof of this case is somehow sketched since the reasoning is similar to Case A.

By Prop. IV. 3, the perturbed trajectory $\mathbf{x}^*([0, \bar{t}_2 + d_I], \mathbf{x}_0^*)$ either intersects the set $\bar{\mathcal{M}}$ at an instant \bar{t}_1^* with $|\bar{t}_1^* - \bar{t}_2| < \alpha$ or intersects $\bar{\mathcal{M}}$ at the instants \bar{t}_1^* and \bar{t}_2^* , with $\bar{t}_1^* < \alpha$, and $|\bar{t}_2^* - \bar{t}_2| < \alpha$. Now, for $0 < T < \bar{t}_2$, choose $\alpha_1 > 0$ such as $\alpha_1 < T < \bar{t}_2 - \alpha_1$, and analyze the two possibilities: i) if $|\bar{t}_1^* - \bar{t}_2| < \alpha_1$ then $T < \bar{t}_1^*$ and a case similar to Cases A.1.1 and A.2.1 is obtained; that is, $\|\mathbf{x}_0^* - \mathbf{x}_0\| < \delta_1 = \min\{\beta_2(\epsilon_0, 0, T), \beta_4(\alpha_1)\} \Rightarrow \|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq \epsilon_0$; ii) if $\bar{t}_1^* < \alpha_1$ and $|\bar{t}_2^* - \bar{t}_2| < \alpha_1$ then $0 \leq \bar{t}_1^* < T < \bar{t}_2^*$:

- $0 \leq t \leq \bar{t}_1^*$: $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta_1 \Rightarrow \|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq \epsilon_0$.
- $\bar{t}_1^* < t \leq T$: Here $\|\mathbf{x}^*(\bar{t}_1^{*+}) - \mathbf{x}(\bar{t}_1^{*+})\| = \|A_R \mathbf{x}^*(\bar{t}_1^*) - \mathbf{x}(\bar{t}_1^*)\| \leq \|\mathbf{x}^*(\bar{t}_1^*) - \mathbf{x}(\bar{t}_1^*)\| \leq \epsilon_0 = \beta_2(\epsilon_1, 0, T) \Rightarrow \|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq \epsilon_1$

As a result, in any case $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta_1 \Rightarrow \|\mathbf{x}^*(t) - \mathbf{x}(t)\| \leq \epsilon_1$, and following a reasoning similar to Case A it results that $\|\mathbf{x}_0^* - \mathbf{x}_0\| \leq \delta_1 \Rightarrow d_H(\mathbf{x}^*([0, T]), \mathbf{x}([0, T])) < \epsilon_1$, for $T \in [0, \bar{t}_2)$.

□

D. Relaxed conditions for continuous dependence on the initial condition

In general, it may be hard to check for \mathbf{x}_0 the condition $CA\mathbf{x}(\bar{t}_k, \mathbf{x}_0) \neq 0$, $k = 1, 2, \dots$, since except in the case of low order reset systems (e. g. Example III.1) crossing/reset instants are hard to compute. Some relaxed conditions that have been found to be useful are developed in the following. Consider the set $\bar{\mathcal{M}}_T$, defined as the set of all points in $\bar{\mathcal{M}} \setminus \mathcal{M}_{RU}$ that produce a tangential crossing:

$$\bar{\mathcal{M}}_T := \{\mathbf{x} \in \bar{\mathcal{M}} \setminus \mathcal{M}_{RU} : CA\mathbf{x} = 0\} \quad (56)$$

In addition, a notion of set backward reachability is also needed for the base system. The backward reachable set from $\bar{\mathcal{M}}_T$, $\mathcal{B}(\bar{\mathcal{M}}_T)$, is defined as

$$\mathcal{B}(\bar{\mathcal{M}}_T) := \{\mathbf{x} \in \mathbb{R}^n : \exists t \in \mathbb{R}^+, e^{At}\mathbf{x} \in \bar{\mathcal{M}}_T\} \quad (57)$$

As result, it is easy to check (see Fig. 16 for a representation of the involved sets) that $CA\mathbf{x}(\bar{t}_k, \mathbf{x}_0) \neq 0$, $k = 1, 2, \dots$ if

$$\mathbf{x}_0 \in D := \mathbb{R}^n \setminus (\mathcal{M} \cup \mathcal{B}(\bar{\mathcal{M}}_T)) \quad (58)$$

and

$$\mathcal{M}_R \cap \mathcal{B}(\bar{\mathcal{M}}_T) = \emptyset \quad (59)$$

Moreover, a more relaxed and conservative condition, but even easier to evaluate, is simply that

$$CA\mathbf{x} \neq 0 \text{ for any } \mathbf{x} \in \bar{\mathcal{M}} \quad (60)$$

note that in this case $\mathcal{M}_T = \mathcal{B}(\mathcal{M}_T) = \emptyset$, and thus from (49) it directly follows that $D = \mathbb{R}^n \setminus \mathcal{M}$; and, in addition, (50) is trivially satisfied.

Example IV.5: The reset control system of Example IV.1 depends continuously on the initial condition at $\mathbf{x}_0 \in \mathbb{R}^2 \setminus \mathcal{M}$, since $\bar{\mathcal{M}}_T = \mathcal{N}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \emptyset$. Note that for second order reset control systems with observable base system, it always turns out that $D = \mathbb{R}^2 \setminus \mathcal{M}$.

Example IV.6: For the reset control system of Example IV.2, $\bar{\mathcal{M}}_T = \text{span}\{(0, 0, 1)\} \setminus \{0\}$ and thus a reachability analysis is necessary to determine an initial set D for the reset control system to depend continuously on the initial condition. Note that the more relaxed condition $\bar{\mathcal{M}}_T = \emptyset$ do not apply in this example. Here, the backward reachable set is

$$\mathcal{B}(\bar{\mathcal{M}}_T) = \bigcup_{t \geq 0} \text{span}\{e^{-At}(0, 0, 1)\} \quad (61)$$

Fig. 17 shows $\mathcal{B}(\bar{\mathcal{M}}_T)$, $\bar{\mathcal{M}}_T$ and the after-reset set \mathcal{M}_R . It results that $\mathcal{B}(\bar{\mathcal{M}}_T) \cap \mathcal{M}_R = \emptyset$, and then $D = \mathbb{R}^3 \setminus (\mathcal{M} \cup \mathcal{B}(\bar{\mathcal{M}}_T))$. Although it is not possible to exactly compute the set $\mathcal{B}(\bar{\mathcal{M}}_T)$, a superset of $\mathcal{B}(\bar{\mathcal{M}}_T)$ may be obtained by the union of two polytopes P and \hat{P} , that may be computed by the following method: for some constant $N > 0$, let us define a set of row vectors \mathbf{n}_i , $i = 1, 2, \dots, N$, using

$$\mathbf{n}_i(\theta_i) := (\sin(\phi_i) \quad \cos(\phi_i) \quad \cos(\theta_i)) \quad (62)$$

where $\phi_i = \frac{i\pi}{2N}$, and θ_i is a constant to be determined, $i = 1, \dots, N$. Then, the polytope P is given by

$$P = \bigcap_{i=1}^N \left\{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{n}_i(\theta_i)\mathbf{x} \leq 0 \right\} \quad (63)$$

and \hat{P} is similarly defined using $-\mathbf{n}_i$ instead of \mathbf{n}_i . In order to achieve $\mathcal{B}(\bar{\mathcal{M}}_T) \subset P \cup \hat{P}$ with a tight enclosing, θ_i is maximized subject to:

$$\begin{cases} \mathbf{n}_i(\theta_i)e^{-At} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq 0, \quad \forall t \geq 0 \\ 0 \leq \theta_i \leq \frac{\pi}{2} \end{cases} \quad (64)$$

Fig. 17 shows a solution for $N = 64$.

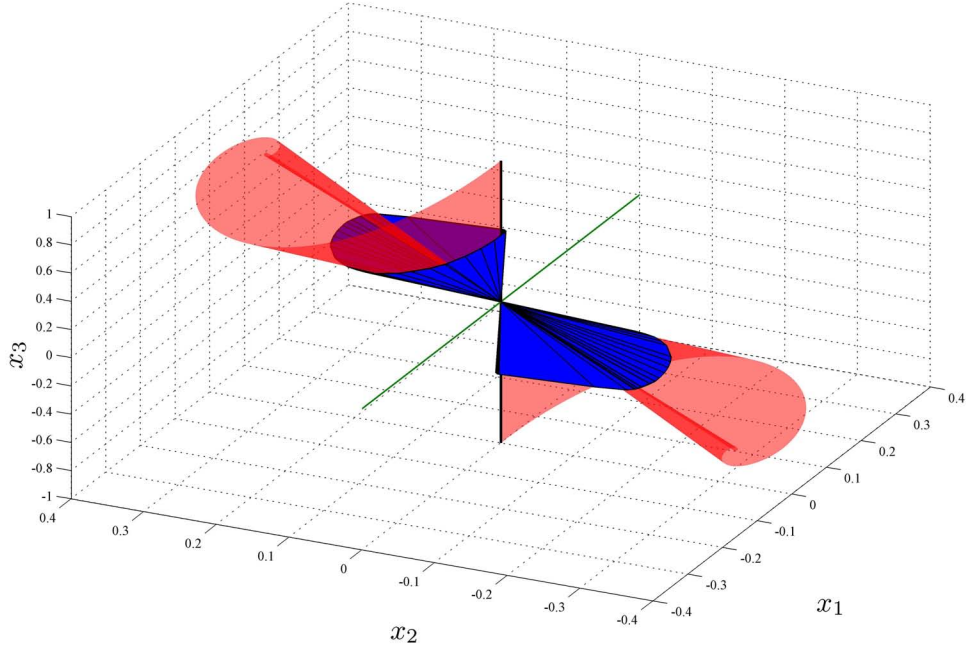


Fig. 17. \mathcal{M}_R (green), $\bar{\mathcal{M}}_T$ (black), $\mathcal{B}(\bar{\mathcal{M}}_T)$ (red), enclosing polytopes P and \hat{P} (blue).

E. Sensitivity of reset control systems to sensor noise

In control practice, the sensitivity to sensor noise is an important issue. It is expected that a reset control system would produce close closed-loop output responses with and without sensor noise, as the sensor noise becomes smaller in some sense. In general, for impulsive and hybrid systems this is a hard issue, and has been one of the main motivation for HI framework [27]. In the following, it will be shown how the property of continuous dependence on the initial condition can be used to analyze sensitivity of a reset control system to sensor noise in the IDS framework, without introducing nondeterminism. It will be assumed that the sensor noise (as well as the other exogenous signals) is a Bohl function, a not overly restrictive condition in practice.

For a reset control system (A, C, n_ρ) , with A and C given by (21), and with state $\mathbf{x} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{x}_p, \mathbf{x}_r)$, a perturbed extended state \mathbf{z}_n is defined as $\mathbf{z}_n = (\mathbf{x}, \mathbf{n})$, where sensor noise is generated by the exosystem (see Fig. 18)

$$\Sigma_n : \begin{cases} \dot{\mathbf{n}}(t) = A_n \mathbf{n}(t), & \mathbf{n}(0) = \mathbf{n}_0 \\ n(t) = C_n \mathbf{n}(t) \end{cases} \quad (65)$$

where $\mathbf{n} \in \mathbb{R}^{m_n}$, $n \in \mathbb{R}$, and A_n, C_n are matrices with appropriate dimensions. In this way, a noisy solution \mathbf{x}^* is recovered as a projection of the noisy extended solution \mathbf{z}_n with initial condition $(\mathbf{x}_0, \mathbf{n}_0)$, for some $\mathbf{n}_0 \neq \mathbf{0}$, that is $\mathbf{x}^* = \Pi \mathbf{z}_n$. Moreover, the noisy control system will be referred to as (A_z, C_z, n_ρ) , where matrices A_z and C_z can be easily obtained. And the noise-free solution \mathbf{x} is simply $\mathbf{x} = \Pi \mathbf{z}$, where \mathbf{z} is the extended solution with initial condition $(\mathbf{x}_0, \mathbf{0})$.

Definition IV.2: A reset control system (A, C, n_ρ) with well-posed reset instants, and with initial condition \mathbf{x}_0 , is not sensitive to noise if for any exosystem Σ_n , any $\epsilon > 0$, and for almost any $T > 0$ there exist $\delta > 0$ such that $\|\mathbf{n}_0\| < \delta \Rightarrow d_H(\Pi \mathbf{z}_n([0, T], (\mathbf{x}_0, \mathbf{n}_0)), \mathbf{x}([0, T], \mathbf{x}_0)) < \epsilon$.

In the following, it will be shown how continuous dependence on the initial condition results in that a reset control system is not sensitive to noise. Since an easily checkable condition is wanted, the result is particularized for full-reset/right-reset compensation and the relaxed condition (49)-(50) (obviously a

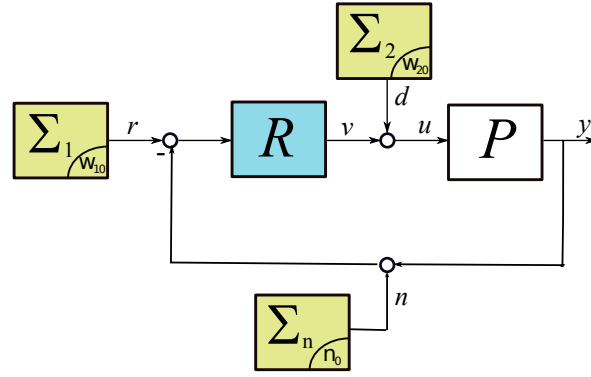


Fig. 18. Reset control system with added sensor noise in the feedback loop.

more relaxed and conservative condition is (51)).

Proposition IV.5: A reset control system (A, C, n_ρ) , with a full reset or right reset compensator, and with initial condition \mathbf{x}_0 , is not sensitive to noise if $\mathbf{x}_0 \in D = \mathbb{R}^n \setminus (\mathcal{M} \cup \mathcal{B}(\bar{\mathcal{M}}_T))$ and $\mathcal{M}_R \cap \mathcal{B}(\bar{\mathcal{M}}_T) = \emptyset$.

Proof: Since the reset compensator is full reset or right reset, then by Prop. III.4 and Prop. IV.1 both the noise-free and the noisy reset control systems have well-posed crossing and reset instants. Let (\bar{t}_k) and (\bar{t}_k^n) , $k = 1, 2, \dots$ be the crossing instants sequences corresponding to the noise-free and the noisy reset control systems, respectively. It turns out that for the initial conditions \mathbf{x}_0 and $(\mathbf{x}_0, \mathbf{0})$, $\bar{t}_k^n = \bar{t}_k$, for $k = 1, 2, \dots$. Now, from (49)-(50) it directly follows that $CA\mathbf{x}(\bar{t}_k, \mathbf{x}_0) \neq 0$, $k = 1, 2, \dots$; and thus for the noisy system (A_z, C_z, n_ρ) (see Fig. 15) it results that

$$\begin{aligned} C_z A_z \mathbf{z}(\bar{t}_k^n, (\mathbf{x}_0, \mathbf{0})) &= \begin{pmatrix} C & -C_n \end{pmatrix} \begin{pmatrix} A & \star \\ O & A_n \end{pmatrix} \begin{pmatrix} \mathbf{x}(\bar{t}_k, \mathbf{x}_0) \\ \mathbf{0} \end{pmatrix} \\ &= CA\mathbf{x}(\bar{t}_k, \mathbf{x}_0) \neq 0 \end{aligned} \quad (66)$$

and then directly by Prop. IV.4 the solution of the noisy reset control system (A_z, C_z, n_ρ) depends continuously on the initial condition at $\mathbf{z}_0 = (\mathbf{x}_0, \mathbf{0})$. As a result, it is true that for any $\epsilon > 0$ and almost any $T \in \mathbb{R}^+$ there exist a $\delta > 0$ such as for any perturbed initial condition $(\mathbf{x}_0, \mathbf{n}_0) \in \mathbb{R}^{n+m_n}$, with $\|\mathbf{n}_0\| < \delta$ it is satisfied that $d_H(\mathbf{z}_n([0, T], (\mathbf{x}_0, \mathbf{n})), \mathbf{z}([0, T], (\mathbf{x}_0, \mathbf{0}))) < \epsilon$ and the result directly follows. \square

Example IV.7 : Consider a reset control system $(A, C, 1)$ as given by Fig. 15, Where R is a P+CI compensator, a parallel connection of a proportional compensator and a CI (see [9] for a detailed definition), and the plant P is an integrator. P+CI is given by matrices $A_r = 0$, $B_r = 1$, $C_r = K_{CI}$, $D_r = K_P$, $A_\rho = 0$, where $K_P = 2$ and $K_{CI} = 1$ for this example; and, in addition, the exogenous signal r is a step of height w_{10} (no disturbance is considered in this example). In this case, A and C are given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \quad (67)$$

and the after-reset and reset sets are $\mathcal{M}_R = \text{span}\{(1, 1, 0)\}$ and $\mathcal{M} = \mathcal{H}_C \setminus \mathcal{M}_R$, respectively, where \mathcal{H}_C is the hyperplane $\mathcal{H}_C = \text{span}\{(1, 1, 0), (0, 0, 1)\}$. Note that P+CI is a full reset compensator, and in addition $\mathcal{M}_{RU} = \mathcal{M}_R = \text{span}\{(1, 1, 0)\}$, and

$$\bar{\mathcal{M}}_T = \mathcal{N}\left(\begin{pmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \end{pmatrix}\right) \setminus \text{span}\{(1, 1, 0)\} = \emptyset \quad (68)$$

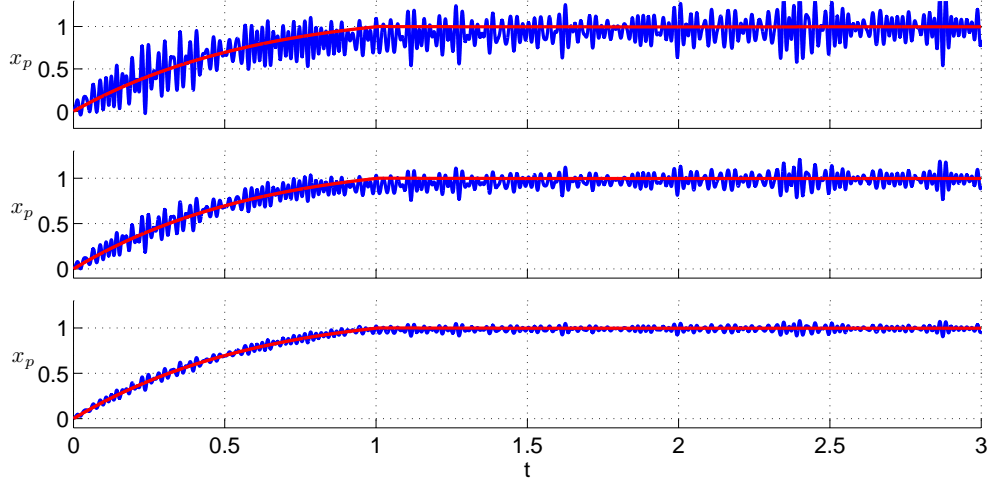


Fig. 19. Closed-loop output (x_p): (blue) noisy case for different values of $\|\mathbf{n}_0\|$, (red) Noise-free case.

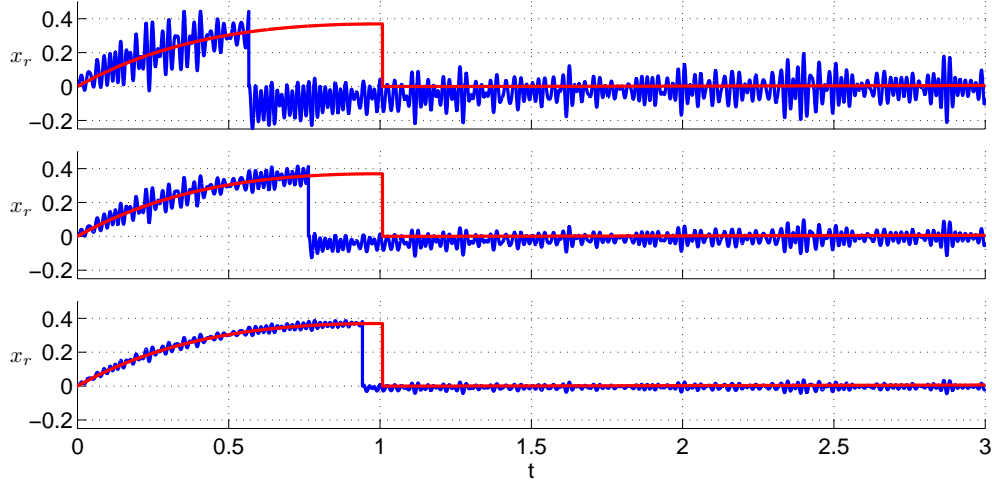


Fig. 20. CI state (x_r): (blue) noisy case for different values of $\|\mathbf{n}_0\|$ —there are many reset actions, with the first reset getting closer to $t = 1$ as $\|\mathbf{n}_0\|$ is getting smaller, (red) Noise-free case—there is only a reset action at $t = 1$.

Thus, by Prop. IV.5, $(A, C, 1)$ is not sensitive to noise for any initial condition $\mathbf{x}_0 \in D = \mathbb{R}^n \setminus \mathcal{M}$. Fig. 19-20 show a time simulation, including closed-loop output $y(= x_p)$ and the reset compensator state x_r , for a noise signal generated by an exosystem Σ_n with different values of \mathbf{n}_0 : it is given by the sum of 20 sinusoidal signals with frequencies greater than 200 rad/s. The reference is a unit step and the CI and the plant are initially at rest, that is $\mathbf{x}_0 = (1, 0, 0) \in D$.

Example IV.8 : Consider the reset control system of Example IV.6, in which $\mathcal{M}_R \cap \mathcal{B}(\bar{\mathcal{M}}_T) = \emptyset$ (see Fig. 14). The after-reset and reset sets are $\mathcal{M}_R = \text{span}\{(1, 0, 0)\}$ and $\mathcal{M} = \text{span}\{(1, 0, 0), (0, 0, 1)\} \setminus \mathcal{M}_R$, respectively. Thus, Prop. IV.5 applies and this reset control system is not sensitive to noise for any initial condition $\mathbf{x}_0 \in D = \mathbb{R}^n \setminus \mathcal{M} \cup \mathcal{B}(\bar{\mathcal{M}}_T)$, where the set $\mathcal{B}(\bar{\mathcal{M}}_T)$ can be bounded by two polytopes. Fig. 21 shows the solution of the reset control system for $\mathbf{x}_0 = (1, 0, 0) \in D$ and with a noise signal n generated as in Example IV.7. Again, this simulation (jointly with many others) reflects the property of reset control systems to be not sensitive to noise according to Def. IV.2 and Prop. IV.5.

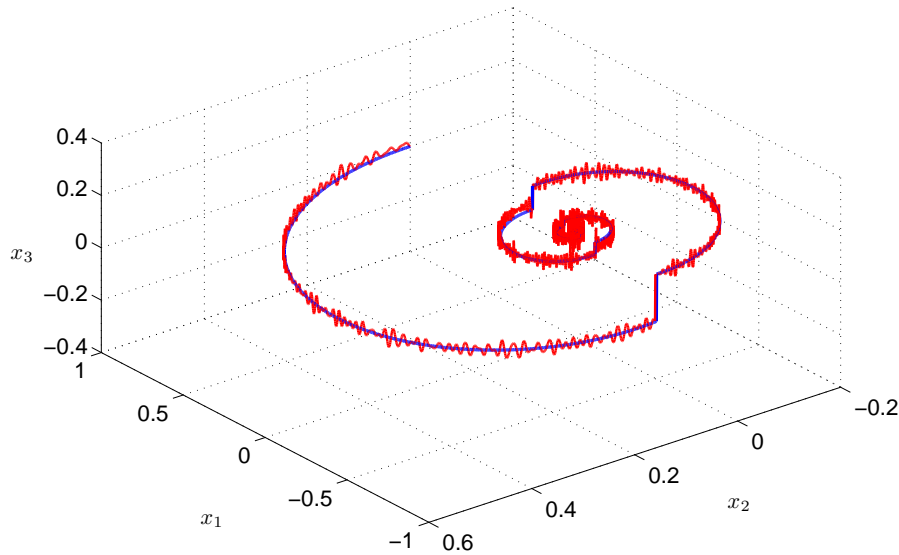


Fig. 21. Solutions of the reset control system-Ex. IV.8: (blue) noise-free case, (red) noisy case.

V. CONCLUSIONS

Well-posedness of reset control systems, that is existence and uniqueness of solutions and continuous dependence on the initial condition, has been investigated in an impulsive dynamical systems framework. Necessary and sufficient conditions for existence and uniqueness of solutions, and a sufficient condition for continuous dependence on the initial condition have been obtained. It turns out that reset compensators that have been successfully used in practice (full reset and right reset compensators) result in well-posed reset control systems, as far as the plant is strictly proper and exogenous signals are represented by Bohl functions. An immediate consequence is that time regulation is not needed for avoiding Zeno solutions (in fact there is no Zeno solutions), and that the reset control system is not sensitive to sensor noise once the continuous dependence property is satisfied. This work has been centered in reset control system with a zero-crossing resetting law. It is believed that the different concepts and methods that have been developed will provide a solid IDS framework to analyze several others resetting laws that has been found useful in practice.

REFERENCES

- [1] W. Aangenent, G. Witvoet, W. Heemels, M. van de Molengraft, and M. Steinbuch, "Performance analysis of reset control systems", *Int. J. of Robust and Nonlinear Control*, **20**, 11, pp. 1213-1233, 2010.
- [2] A. Abate, A. D'Innocenzo, M. D. Di Benedetto, S. Sastry, "Understanding deadlock and livelock behaviors in hybrid control systems", *Nonlinear Analysis: Hybrid Systems*, **3**, pp. 150-162, 2009.
- [3] B. Ahmad, S. Sivasundaram, "The monotone iterative technique for impulsive hybrid set valued integro-differential equations", *Nonlinear Analysis*, **65**, 2, pp. 2260-2276, 2006.
- [4] P. J. Antsaklis, "A brief introduction to the theory and applications of hybrid systems", *Proceedings of the IEEE*, **88**, 7, pp. 879-887, 2000.
- [5] D. D. Bainov, P. S. Simeonov, *Systems with impulse effect: stability, theory and applications*, Ellis Horwood Limited, Chichester, 1989.
- [6] A. Baños, A. Barreiro, "Delay-independent stability of reset systems", *IEEE Trans. Automatic Control*, **54**, 2, pp. 341-346, 2009.
- [7] A. Baños, J. Carrasco, A. Barreiro, "Reset times dependent stability of reset systems", *IEEE Trans. Automatic Control*, **56**, 1, pp. 217-223, 2011.
- [8] A. Baños, J. I. Mulero, "Well-posedness of reset control systems as state-dependent impulsive dynamical systems", *Abstract and Applied Analysis*, vol. 2012, doi:10.1155/2012/808290, 2012.
- [9] A. Baños, A. Barreiro (2012), *Reset Control Systems*, AIC Series, Springer, London, 2012.
- [10] A. Barreiro, A. Baños, "Delay-dependent stability of reset systems", *Automatica*, **46**, 1, pp. 216-221, 2010.
- [11] A. Barreiro, A. Baños, S. Dormido, and J. A. González-Prieto, "Reset control systems with reset band: well-posedness, limit cycles and stability analysis", *Systems and Control Letters*, **63**, pp. 1-11, 2014.
- [12] M. S. Branicky, *Studies in Hybrid Systems: Modeling, Analysis, and Control*, Ph. D. Thesis, M.I.T., 1995.
- [13] O. Beker, *Analysis of reset control systems*, Ph. D. Thesis, University of Massachusetts Amherst, 2001.

- [14] O. Beker, C.V. Hollot, Y. Chait, and H. Han, "Fundamental properties of reset control systems". *Automatica*, **40**, pp.905-915, 2004
- [15] Broucke, M., A. Arapostathis, "Continuous selections of trajectories of hybrid systems", *Systems and Control Letters*, **47**, pp. 149-157, 2002.
- [16] Cai, C., Goebel, R., Teel, A. R., "Relaxion results for hybrid inclusions", *Set-Vaued Analysis*, **16**, pp. 733-757, 2008.
- [17] M. K. Camlibel, W. P. M. H. Heemels, A. J. van der Schaft, and J. M. Schumacher, "Solutions concepts for hybrid dynamical systems", *IFAC World Congress*, Barcelona, 2002.
- [18] J. Carrasco, *Stability of reset control systems*, Ph. D. Thesis (in spanish), University of Murcia, 2009.
- [19] J. Carrasco, A. Baños, A. J. van der Schaft, "A passivity-based approach to reset control system stability", *Systems and Control Letters*, **59**, 1, pp. 18-24, 2010.
- [20] J. Carrasco, A. Baños, "Reset control of an industrial in-line pH process", *IEEE Transactions on Control Systems Technology*, **20**, 4, pp. 1100-1106, 2012.
- [21] J. C. Clegg, "A nonlinear integrator for servomechnisms", *Transactions A.I.E.E.m, Part II*, **77**, pp. 41-42, 1958.
- [22] D. A. Copp and R. G. Sanfelice, "On the effect and robustness of zero-crossing detection algorithms in simulation of hybrid systems jumping on surfaces", *Proc. American Control Conference*, pp. 2449-2454, 2012.
- [23] A. Dishliev, K. Dishlieva, S. Nenov, *Specific asymptotic properties of the solutions of impulsive differential equations: methods and applications*, Academic Publications, 2012.
- [24] E. Garone, R. Naldi, E. Frazzoli, "Switching control laws in the presence of measurement noise", *Systems and Control Letters*, **59**, pp-353-364, 2010.
- [25] I. Gohberg, P. Lancaster, and L. Rodman, *Invariant subspaces of matrices with applications*, Classics in Applied Mathematics, SIAM, Philadelphia, 2006.
- [26] R. Goebel, R. S. Sanfelice, and A. R. Teel, "Hybrid dynamical systems", *IEEE Control Systems Magazine*, **29**, 2, pp- 28-93, 2009.
- [27] R. Goebel, R. S. Sanfelice, and A. R. Teel, *Hybrid dynamical systems: modelling, stability and robustness*, Princeton Univ. Press, 2012.
- [28] W. M. Haddad, V. Chellaboina, V., and S. G. Nersesov, *Impulsive and hybrid dynamical systems: stability, dissipativity, and control*, Princeton University Press, 2006.
- [29] T. Yucelen, W. M. Haddad, "Consensus protocols for networked multi-agent systems with a uniformly continuous quasi-resetting architecture", *International Journal of Control*, **87**, 8, pp. 1716-1727, 2014.
- [30] I. M. Horowitz and P. Rosenbaum, "Nonlinear design for cost of feedback reduction in systems with large parameter uncertainty", *International Journal of Control*, **24**, 6, pp. 977-1001, 1975.
- [31] K. R. Krishnan and I. M. Horowitz, "Synthesis of a nonlinear feedback system with significant plant-ignorance for prescribed system tolerances", *International Journal of Control*, **19**, 4, pp. 689-706, 1974.
- [32] W. P. M. H. Heemels, M. K. Camlibel, A. J. van der Schaft, and J. M. Schumaker, "On the existence and uniqueness of solutions to hybrid dynamical systems", in R. Johansson and A. Rantzer (eds.), *Nonlinear and hybrid control in automotive applications*, Springer, London, 2003.
- [33] J. P. Hespanha, *Linear Systems Theory*. Princeton, New Jersey: Princeton Press, 2009.
- [34] J. Hadamard, "Sur les problèmes aux dérivées partiales et leur signification physique", *Princeton University Bulletin*, pp. 49-52, 1902.
- [35] J. Imura, A. J. van der Schaft, "Characterization of well-posedness of piecewise-linear systems", *IEEE Trans. Automatic Control*, **45**, pp. , 2000.
- [36] V. Lakshmikanthan, D. Bainov., P. S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.
- [37] J. Lunze, F. Lamnabhi-Lagarigue (Eds.), *Handbook of hybrid systems control*, Cambridge University Press, Cambridge, 2009.
- [38] J. Lygeros, K. H. Johansson, S. N. Simić, J. Zhang, and S. S. Sastry, "Dynamical properties of hybrid automata", *IEEE Transactions on Automatic Control*, **48**, 1, pp. 2-16, 2003.
- [39] S. J. L. M. van Loon, B. G. B. Hunnekens, W. P. M. H. Heemels, N. van de Wouw, H. Nijmeijer, "Transient improvement of linear systems using a split-path nonlinear integrator", *American Control Conference*, pp. 341-346, 2014.
- [40] A. Michel, L. Hou, D. Liu, *Stability of dynamical systems: continuous, discontinuous, and discrete systems*, Birkauer, Boston, 2007
- [41] D. Nesic, L. Zaccarian, A. R. Teel, "Stability properties of reset systems", *Automatica*, **44**, pp. 2019-2026, 2008.
- [42] D. Nesic, A. R. Teel, L. Zaccarian, "Stability and performance of SISO control systems with first order reset elements", *IEEE Trans. Automatic Control*, **56**, pp. 2567-2582, 2011.
- [43] S. Polenkova, J. Polderman, R. Langerak, "Stability of reset systems", *International Symp. Mathematical Theory of Networks and Systems*, Melbourne, Australia, 2012.
- [44] B. de Schutter, W. P. M. H. Heemels, J. Lunze, and C. Prieur, "Survey of modeling, analysis, and control of hybrid systems", in J. Lunze, F. Lamnabhi-Lagarigue (Eds.), *Handbook of hybrid systems control*, Cambridge University Press, Cambridge, pp. 33-55, 2009.
- [45] A. R. Teel, R. G. Sanfelice, R. Goebel, "Hybrid Control Systems", in R. A. Meyers (Ed.), *Mathematics of complexity and dynamical systems*, Springer, pp. 704-728, 2012.
- [46] H. L. Trentelman, A. A. Stoorvogel, M. Hautus, *Control theory for linear systems*, Springer, London, 2001.
- [47] A. Vidal and A. Baños, "Reset compensation for temperature control: experimental application on heat exchangers", *Chemical Engineering Journal*, **159**, 1-3, pp. 170-181, 2010.
- [48] M. Vidyasagar, *Nonlinear systems stability*, Prentice-Hall, London, 1993.
- [49] T. Yang, *Impulsive control theory*, Lectures Notes in Control and Information Science 272, Springer, Berlin, 2001.

APPENDIX

In the following, some technical results will be derived for the left reset control system of Section IV.C (LRC system in the following). In general, including single-input single-output systems, it is not true that for an arbitrary system (A, B, C) the geometric multiplicity of unobservable modes is 1. However, for the

LRC base system this is indeed the case. On the other hand, in this left compensation case, the subspace of after reset and unobservable states, \mathcal{M}_{RU} , is A -invariant only in the simple case in which it consists of the zero state. In addition, the unobservable subspace shows a particular structure, it can be expressed through the root spaces of the unobservable modes.

Notation and Background: $\mathcal{R}_\lambda(A)$ represents the root space of A associated to $\lambda \in \sigma(A)$. \oplus stands for direct sum; V is a A -cyclic subspace of \mathbb{R}^n generated by \mathbf{v} if $V = \text{span}\{\mathbf{v}, A\mathbf{v}, \dots\}$. Assume that λ has algebraic multiplicity ma_λ and geometric multiplicity 1, and let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{ma_\lambda}\}$ be the set of generalized eigenvectors (including the eigenvector \mathbf{v}_1), then $\mathcal{R}_\lambda(A) = \mathcal{N}(\lambda I - A)^{ma_\lambda} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{ma_\lambda}\}$; therefore, if the mode $\lambda \in \sigma(A)$ is observable then $\mathbf{v}_i \notin \mathcal{N}(\mathcal{O})$, for $i = 1, \dots, n$ (note that $C\mathbf{v}_1 \neq 0$ by the PBH test $\Rightarrow \mathbf{v}_1 \notin \mathcal{N}(\mathcal{O})$; $CA\mathbf{v}_2 = \lambda C\mathbf{v}_2 + C\mathbf{v}_1 \Rightarrow C\mathbf{v}_2 \neq 0$ or $CA\mathbf{v}_2 \neq 0 \Rightarrow \mathbf{v}_2 \notin \mathcal{N}(\mathcal{O})$, \dots).

Proposition A.1: For the LRC system, the geometric multiplicity of any $\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)$, as an eigenvalue of A , is 1.

Proof: By using the PBH test, for any $\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)$ it is true that

$$\begin{pmatrix} \lambda I - \bar{A}_p & O & -\bar{B}_p C_{r_2} \\ B_{r_1} \bar{C}_p & \lambda I - A_{r_1} & O \\ O & -B_{r_2} C_{r_1} & \lambda I - A_{r_2} \\ \bar{C}_p & O & O \end{pmatrix} \begin{pmatrix} \bar{\mathbf{v}}_p \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (69)$$

for some nonzero state $(\bar{\mathbf{v}}_p, \mathbf{v}_1, \mathbf{v}_2)$ and thus

$$\begin{pmatrix} \lambda I - \bar{A}_p & -\bar{B}_p \\ \bar{C}_p & O \end{pmatrix} \begin{pmatrix} \bar{\mathbf{v}}_p \\ C_{r_2} \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (70)$$

$$\begin{aligned} (\lambda I - A_{r_1})\mathbf{v}_1 &= \mathbf{0} \\ (\lambda I - A_{r_2})\mathbf{v}_2 &= B_{r_2} C_{r_1} \mathbf{v}_1 \end{aligned}$$

Now, λ is a zero of \bar{P} and an eigenvalue of A_{r_1} and/or A_{r_2} ; and in any case, the geometric multiplicity of the zero and the eigenvalues is 1, since the realizations are minimal. As a result, two cases are possible: i) λ is not an eigenvalue of A_{r_1} and is an eigenvalue of A_{r_2} , thus $\mathbf{v}_1 = \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$ with $\dim(\mathcal{N}(\lambda I - A_{r_2})) = 1$, ii) λ is an eigenvalue of A_{r_1} , thus $\mathbf{v}_1 \neq \mathbf{0}$ with $\dim(\mathcal{N}(\lambda I - A_{r_1})) = 1$, and $\mathbf{v}_2 = \mathbf{0}$ or $\mathbf{v}_2 \neq \mathbf{0}$ with $\dim(\mathcal{N}(\lambda I - A_{r_2})) = 1$. As a result, the geometric multiplicity of λ as an eigenvalue of A is 1. \square

Proposition A.2: For the LRC system, \mathcal{M}_{RU} is A -invariant if and only if $\mathcal{M}_{RU} = \{\mathbf{0}\}$.

Proof: (if) If $\mathcal{M}_{RU} = \{\mathbf{0}\}$ then \mathcal{M}_{RU} is trivially A -invariant. (only if) If \mathcal{M}_{RU} is A -invariant then $E_\lambda(A) \cap \mathcal{M}_{RU} \neq \{\mathbf{0}\}$ for some $\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)$, that is \mathcal{M}_{RU} contains at least one eigenvector, or $\mathcal{M}_{RU} = \{\mathbf{0}\}$. By using the PBH test for observability (69), for any eigenvector $\mathbf{v} = (\bar{\mathbf{v}}_p, \mathbf{v}_{\bar{p}}, \mathbf{v}_\rho)$ with $\mathbf{v}_\rho = \mathbf{0}$ it results that $(\bar{A}_p - \lambda I)\bar{\mathbf{v}}_p = \mathbf{0}$ and thus $\bar{\mathbf{v}}_p = \mathbf{0}$ since λ is not an eigenvalue of \bar{A}_p (otherwise, the realization $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$ would not be minimal since λ is a zero). In addition, from (70) it results that $(A_{r_1} - \lambda I)\mathbf{v}_{\bar{p}} = \mathbf{0}$ and $B_{r_2} C_{r_1} \mathbf{v}_{\bar{p}} = \mathbf{0}$. Now, since both $(A_{r_1}, B_{r_1}, C_{r_1})$ and $(A_{r_2}, B_{r_2}, C_{r_2})$ are minimal it is true that $B_{r_2} \neq \mathbf{0}$ and $C_{r_1} \mathbf{v}_{\bar{p}} \neq 0$ (scalar) for $\mathbf{v}_{\bar{p}} \neq \mathbf{0}$, then it must be true that $\mathbf{v}_{\bar{p}} = \mathbf{0}$ and thus $E_\lambda(A) \cap \mathcal{M}_{RU} = \{\mathbf{0}\}$ for any $\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)$. As a result, \mathcal{M}_{RU} does not contain any eigenvector and thus $\mathcal{M}_{RU} = \{\mathbf{0}\}$. \square

Proposition A.3: The unobservable subspace of the LRC system is given by

$$\mathcal{N}(\mathcal{O}) = \bigoplus_{\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)} \mathcal{N}(\lambda I - A)^{d_\lambda} \quad (71)$$

Proof: Consider some $\lambda \in \sigma_{\bar{\mathcal{O}}}(A, C)$. Firstly, since λ is a zero of $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$ with algebraic multiplicity $m_\lambda \geq d_\lambda$ (d_λ is the number of cancellations) then $\bar{C}_p(\lambda I - \bar{A}_p)^{-j} \bar{B}_p = 0$, for $j = 0, \dots, d_\lambda$. In addition, λ is an eigenvalue of A with geometric multiplicity 1, and algebraic multiplicity $ma_\lambda \geq d_\lambda$. Thus, it is possible to choose a subset $\{\mathbf{v}_\lambda^{(d_\lambda-k)} = (\mathbf{v}_{p_\lambda}^{(d_\lambda-k)}, \mathbf{v}_{\bar{p}_\lambda}^{(d_\lambda-k)}, \mathbf{v}_{\rho_\lambda}^{(d_\lambda-k)})\}$, $k = 0, \dots, d_\lambda - 1$ of its generalized eigenvalues (including the eigenvector corresponding to $k = 0$), that it is a basis of $\mathcal{N}(\lambda I - A)^{d_\lambda}$. By

using again the PBH test, it is clear from (42)-(43) that $\mathbf{v}_{p_\lambda}^{(d_\lambda)} = (\lambda I - \bar{A}_p)^{-1} \bar{B}_p C_{r_2} \mathbf{v}_{\rho_\lambda}^{(d_\lambda)}$, and in general $\mathbf{v}_{p_\lambda}^{(d_\lambda-k)} = \sum_{j=1}^{k+1} (\lambda I - \bar{A}_p)^{-j} \bar{B}_p C_{r_2} \mathbf{v}_{\rho_\lambda}^{(d_\lambda+j-(k+1))}$ for $k = 0, \dots, d_\lambda - 1$. Note that the inverse is well-defined in all cases since $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$ is observable and λ is a zero of $(\bar{A}_p, \bar{B}_p, \bar{C}_p)$ with multiplicity at least d_λ . For the eigenvector $\mathbf{v}_\lambda^{(d_\lambda)} = (\mathbf{v}_{p_\lambda}^{(d_\lambda)}, \mathbf{v}_{\bar{\rho}_\lambda}^{(d_\lambda)}, \mathbf{v}_{\rho_\lambda}^{(d_\lambda)})$ it has been shown that the scalar $C_{r_2} \mathbf{v}_{\rho_\lambda}$ is not equal to zero,

As a result, it is true that $C \mathbf{v}_\lambda^{(d_\lambda-k)} = \sum_{j=1}^{k+1} \bar{C}_p (\lambda I - \bar{A}_p)^{-j} \bar{B}_p C_{r_2} \mathbf{v}_{\rho_\lambda}^{(d_\lambda+j-(k+1))} = 0$, for any $k = 0, \dots, d_\lambda - 1$. Thus, since by construction the subspace $\mathcal{N}(\lambda I - A)^{d_\lambda}$ is A -invariant it follows that $C A^j \mathbf{v}_\lambda^{(d_\lambda-k)} = 0$ for $j = 0, 1, \dots$, and $k = 0, 1, \dots, d_\lambda - 1$; in other words, $\mathcal{N}(\lambda I - A)^{d_\lambda} \subset \mathcal{N}(\mathcal{O})$.

In addition, since the algebraic multiplicity of λ , as eigenvalue of A , is greater than d_λ only when $m_\lambda = d_\lambda$ then it follows that in this case $\bar{C}_p (\lambda I - \bar{A}_p)^{-j} \bar{B}_p \neq 0$, for $j > d_\lambda$; and thus it is not difficult to see that $\mathcal{N}(\lambda I - A)^{d_\lambda+n} \cap \mathcal{N}(\mathcal{O}) = \mathcal{N}(\lambda I - A)^{d_\lambda}$, for any integer $n > 0$; and, in particular $\mathcal{R}_\lambda(A) \cap \mathcal{N}(\mathcal{O}) = \mathcal{N}(\lambda I - A)^{d_\lambda}$.

Now, consider an observable mode $\lambda \in \sigma_O(A, C)$ with algebraic multiplicity ma_λ . Its root space $\mathcal{R}_\lambda(A)$ is spanned by the set of generalized eigenvectors $\{\mathbf{v}_\lambda^{(d_\lambda-k)} = (\mathbf{v}_{p_\lambda}^{(d_\lambda-k)}, \mathbf{v}_{\bar{\rho}_\lambda}^{(d_\lambda-k)}, \mathbf{v}_{\rho_\lambda}^{(d_\lambda-k)})\}$, $k = 0, \dots, ma_\lambda - 1$, and it is true that $\mathbf{v}_\lambda^{(d_\lambda-k)} \notin \mathcal{N}(\mathcal{O})$, for any $k = 0, \dots, ma_\lambda - 1$. As a result, since no generalized eigenvector is an element of the unobservable subspace, then it follows that no A -cyclic subspace must be in the unobservable subspace, and thus $\mathcal{R}_\lambda(A) \cap \mathcal{N}(\mathcal{O}) = \{\mathbf{0}\}$ for observable modes.

Finally, since $\mathcal{N}(\mathcal{O})$ is A -invariant, it may be obtained as the direct sum of its intersection with the root spaces, and thus (43) directly follows:

$$\mathcal{N}(\mathcal{O}) = \bigoplus_{\lambda \in \sigma(A)} \mathcal{R}_\lambda(A) \cap \mathcal{N}(\mathcal{O}) = \bigoplus_{\lambda \in \sigma_{\mathcal{O}}(A, C)} \mathcal{N}(\lambda I - A)^{d_\lambda}$$

□